In this unit, students study arithmetic and geometric sequences and implicit and explicit rules for defining them. Then they analyze exponential and logarithmic patterns and graphs as well as properties of logarithms. Finally, they solve exponential and logarithmic equations.

Vocabulary Development
The key terms for this unit can be found on the Unit Opener page. These terms are divided into Academic Vocabulary and Math Terms. Academic Vocabulary includes terms that have additional meaning outside of math. These terms are listed separately to help students transition from their current understanding of a term to its meaning as a mathematics term. To help students learn new vocabulary:

- Have students discuss meaning and use graphic organizers to record their understanding of new words.
- Remind students to place their graphic organizers in their math notebooks and revisit their notes as their understanding of vocabulary grows.
- As needed, pronounce new words and place pronunciation guides and definitions on the class Word Wall.

Embedded Assessments
Embedded assessments allow students to do the following:

- Demonstrate their understanding of new concepts.
- Integrate previous and new knowledge by solving real-world problems presented in new settings.

They also provide formative information to help you adjust instruction to meet your students’ learning needs.

Prior to beginning instruction, have students unpack the first embedded assessment in the unit to identify the skills and knowledge necessary for successful completion of that assessment. Help students create a visual display of the unpacked assessment and post it in your class. As students learn new knowledge and skills, remind them that they will be expected to apply that knowledge to the assessment. After students complete each embedded assessment, turn to the next one in the unit and repeat the process of unpacking that assessment with students.

**AP / College Readiness**
Unit 4 continues to develop students’ understanding of functions and their inverses by:
- Graphing exponential and logarithmic functions.
- Applying properties of exponents to develop properties of logarithms.
- Solving exponential and logarithmic equations.

**Unpacking the Embedded Assessments**
The following are the key skills and knowledge students will need to know for each assessment.

**Embedded Assessment 1**
**Sequences and Series, The Chessboard Problem**
- Identifying terms in arithmetic and geometric sequences
- Identifying common differences and common ratios
- Writing implicit and explicit rules for arithmetic and geometric sequences

**Embedded Assessment 2**
**Exponential Functions and Common Logarithms, Whether or Not**
- Examining exponential patterns and functions
- Identifying and analyzing exponential graphs
- Transforming exponential functions
- Graphing and transforming natural base exponential functions
- Examining common logarithmic functions
- Understanding properties of logarithms

**Embedded Assessment 3**
**Exponential and Logarithmic Equations, Evaluating Your Interest**
- Solving exponential equations
- Solving logarithmic equations
- Solving real-world applications of exponential and logarithmic functions
**Suggested Pacing**
The following table provides suggestions for pacing using a 45-minute class period. Space is left for you to write your own pacing guidelines based on your experiences in using the materials.

<table>
<thead>
<tr>
<th>45-Minute Period</th>
<th>Your Comments on Pacing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit Overview/Getting Ready</td>
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<tr>
<td>Activity 19</td>
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<td>Activity 20</td>
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<tr>
<td>Embedded Assessment 1</td>
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<td>Activity 22</td>
<td>4</td>
</tr>
<tr>
<td>Embedded Assessment 2</td>
<td>1</td>
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<td>Activity 23</td>
<td>3</td>
</tr>
<tr>
<td>Activity 24</td>
<td>4</td>
</tr>
<tr>
<td>Embedded Assessment 3</td>
<td>1</td>
</tr>
<tr>
<td><strong>Total 45-Minute Periods</strong></td>
<td><strong>26</strong></td>
</tr>
</tbody>
</table>

**Additional Resources**
Additional resources that you may find helpful for your instruction include the following, which may be found in the Teacher Resources at SpringBoard Digital.

- Unit Practice (additional problems for each activity)
- Getting Ready Practice (additional lessons and practice problems for the prerequisite skills)
- Mini-Lessons (instructional support for concepts related to lesson content)
Unit Overview
In this unit, you will study arithmetic and geometric sequences and series and their applications. You will also study exponential functions and investigate logarithmic functions and equations.

Key Terms
As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Math Terms
- sequence
- arithmetic sequence
- common difference
- recursive formula
- explicit formula
- series
- partial sum
- sigma notation
- geometric sequence
- common ratio
- geometric series
- finite series
- infinite series
- sum of the infinite geometric series
- exponential function
- exponential decay factor
- exponential growth factor
- asymptote
- logarithm
- common logarithm
- logarithmic function
- natural logarithm
- Change of Base Formula
- exponential equation
- compound interest
- logarithmic equation
- extraneous solution

Essential Questions
- How are functions that grow at a constant rate distinguished from those that do not grow at a constant rate?
- How are logarithmic and exponential equations used to model real-world problems?

Embedded Assessments
This unit has three embedded assessments, following Activities 20, 22, and 24. By completing these embedded assessments, you will demonstrate your understanding of arithmetic and geometric sequences and series, as well as exponential and logarithmic functions and equations.

- Embedded Assessment 1: Sequences and Series p. 321
- Embedded Assessment 2: Exponential Functions and Common Logarithms p. 357
- Embedded Assessment 3: Exponential and Logarithmic Equations p. 383

Developing Math Language
As this unit progresses, help students make the transition from general words they may already know (the Academic Vocabulary) to the meanings of those words in mathematics. You may want students to work in pairs or small groups to facilitate discussion and to build confidence and fluency as they internalize new language. Ask students to discuss new academic and mathematics terms as they are introduced, identifying meaning as well as pronunciation and common usage. Remind students to use their math notebooks to record their understanding of new terms and concepts.

As needed, pronounce new terms clearly and monitor students' use of words in their discussions to ensure that they are using terms correctly. Encourage students to practice fluency with new words as they gain greater understanding of mathematical and other terms.
Getting Ready

Write your answers on notebook paper. Show your work.

1. Describe the pattern displayed by 1, 2, 5, 10, 17, . . .
2. Give the next three terms of the sequence 0, −2, 1, −3, . . .
3. Draw Figure 4, using the pattern below. Then explain how you would create any figure in the pattern.

<table>
<thead>
<tr>
<th>Figure 1</th>
<th>Figure 2</th>
<th>Figure 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
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<tr>
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<td></td>
</tr>
</tbody>
</table>

Sample explanation: Each figure has the same number of columns of dots as the figure number, and the number of rows of dots is always one more than the figure number.

4. Simplify each expression.
   a. \( \frac{6x^2}{y^3} \)
   b. \( 6a^2b^4 \)
   c. \( 2a^3b^8 \)

5. Evaluate the expression.
   \( \frac{3^{27}}{3^{33}} \)

6. Express the product in scientific notation.
   \( (2.9 \times 10^3)(3 \times 10^5) \)

7. Solve the equation for \( x \).
   \( 19 = −8x + 35 \)

8. Write a function \( C(t) \) to represent the cost of a taxicab ride, where the charge includes a fee of $2.50 plus $0.50 for each tenth of a mile \( t \). Then give the slope and \( y \)-intercept of the graph of the function.

Answer Key

1. The numbers increase by consecutive increasing odd numbers.
2. 2, −4, 3
3. Figure 4
   ● ● ● ● ●
   ● ● ● ● ●
   ● ● ● ● ●
   ● ● ● ● ●
   ● ● ● ● ●

Sample explanation: Each figure has the same number of columns of dots as the figure number, and the number of rows of dots is always one more than the figure number.

4. a. \( \frac{36x^4}{y^7} \)
   b. \( 6a^2b^4 \)
   c. \( 2a^3b^8 \)

5. \( 3^4 = 81 \)
6. \( 8.7 \times 10^5 \)
7. \( x = 2 \)
8. \( C(t) = 2.5 + 0.5t; \) slope = 0.5; \( y \)-intercept = 2.5

Getting Ready Practice

For students who may need additional instruction on one or more of the prerequisite skills for this unit, Getting Ready practice pages are available in the Teacher Resources at SpringBoard Digital. These practice pages include worked-out examples as well as multiple opportunities for students to apply concepts learned.
Sequences like the one above are called arithmetic sequences. An arithmetic sequence is a sequence in which the difference of consecutive terms is a constant. The constant difference is called the common difference.

### Hydrocarbons

Hydrocarbons are the simplest organic compounds, containing only carbon and hydrogen atoms. Hydrocarbons that contain only one pair of electrons between two atoms are called alkanes. Alkanes are valuable as clean fuels because they burn to form water and carbon dioxide. The number of carbon and hydrogen atoms in a molecule of the first six alkanes is shown in the table below:

<table>
<thead>
<tr>
<th>Alkane</th>
<th>Carbon Atoms</th>
<th>Hydrogen Atoms</th>
</tr>
</thead>
<tbody>
<tr>
<td>methane</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>ethane</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>propane</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>butane</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>pentane</td>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>hexane</td>
<td>6</td>
<td>14</td>
</tr>
</tbody>
</table>

### Example

1. **Model with mathematics.** Graph the data in the table. Write a function $f$, where $f(n)$ is the number of hydrogen atoms in an alkane with $n$ carbon atoms.

   Any function where the domain is a set of positive consecutive integers forms a sequence. The values in the range of the function are the terms of the sequence. When naming a term in a sequence, subscripts are used rather than traditional function notation. For example, the first term in a sequence would be called $a_1$ rather than $f(1)$.

   Consider the sequence $\{4, 6, 8, 10, 12, 14\}$ formed by the number of hydrogen atoms in the first six alkanes.

   2. **What is $a_1$? What is $a_2$?**
      
      $a_1 = 4; a_2 = 6$

   3. **Find the differences $a_2 - a_1, a_3 - a_2, a_4 - a_3, a_5 - a_4,$ and $a_6 - a_5.$**

      Each difference is 2.

Sequences like the one above are called arithmetic sequences. An arithmetic sequence is a sequence in which the difference of consecutive terms is a constant. The constant difference is called the common difference and is usually represented by $d$.

### Writing Math

If the fourth term in a sequence is 10, then $a_4 = 10$.

Sequences may have a finite or an infinite number of terms and are sometimes written in braces { }.

### Common Core State Standards for Activity 19

- HSA-SSE.B.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.*
- HSF-BF.A.1 Write a function that describes a relationship between two quantities.*
- HSF-BF.A.1a Determine an explicit expression, a recursive process, or steps for calculation from a context.
- HSF-BF.A.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.*

### Activity Standards Focus

In Activity 19, students learn to identify arithmetic sequences and to determine the $n$th term of such sequences using recursive and explicit formulas. They also write formulas for the sum of the terms in arithmetic sequences, known as an arithmetic series, and calculate the $n$th partial sums of arithmetic series.

Finally, they represent arithmetic series using sigma notation and determine the sums. There is a lot of notation in this activity, and students may get lost in the symbols. Encourage students to read carefully, and check often to be sure they can explain the meanings of the formulas and the variables within the formulas.

### Lesson 19-1

**PLAN**

**Pacing:** 1 class period

**Chinking the Lesson**

#1 #2–3 #4–5

Check Your Understanding #9–10 #11–14 #15–16

Example A

Check Your Understanding Lesson Practice

**TEACH**

**Bell-Ringer Activity**

Ask students to describe the pattern and give the next 3 terms of the following sequences.

1. $-16, -14, -12, \ldots$ [The difference between any term and the previous term is 2; $-10, -8, -6$]

2. $6, -2, -10, \ldots$ [The difference between any term and the previous term is $-8$, $-18$, $-26$, $-34$.]

Discuss with students the methods they used to determine the patterns.

### Activating Prior Knowledge, Create Representations, Look for a Pattern, Summarizing, Paraphrasing, Vocabulary Organizer

Students plot data and use their knowledge of linear functions to write a rule for $f(n)$. When debriefing, be sure that students understand that the domain of the function must be discrete because $n$ represents the number of carbon atoms.

### 2–3 Debriefing

Use Item 2 to assess whether students understand the meaning of subscripts. Also, note that finding that the differences are constant in Item 3 will help students attach meaning to the definition of common difference that follows.

### ACTIVITY 19

**Guided**
ACTIVITY 4–5 Look for a Pattern, Debriefing

In Item 4, students must generalize, which is a difficult concept for some students. Direct students to the Math Tip. If students need additional guidance understanding that these expressions represent consecutive terms in the sequence, try asking these questions:

What do $a_n$ and $a_{n+1}$ represent when $n = 1$? When $n = 2$? When $n = 3$?

Item 5 connects the concepts of sequence and common difference back to the opening context and shows that consecutive integers form an arithmetic sequence where $d = 1$. Any sequence where the terms remain constant satisfies this condition.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand concepts related to arithmetic sequences. For the sequence in Item 7, each term is multiplied by 2 to find the next term; be sure students can verbalize why this pattern does not determine an arithmetic sequence.

Answers

6. arithmetic; 5
7. not arithmetic
8. 37, 46, 64

9–10 Activating Prior Knowledge, Create Representations, Look for a Pattern

In Item 9, students use their ability to work with literal equations to solve the formula from Item 4 for $a_{n+1}$. In Item 10, students identify each term using math terminology. In order to more easily compare the recursive formula and the explicit formula, students need to understand that the expression $a_n = a_{n-1} + d$ is equivalent to the expression in Item 9. Elicit from students the fact that $a_n$ is the term that follows $a_{n-1}$ just as $a_{n+1}$ is the term that follows $a_n$.

Developing Math Language

Watch for students who interchange the terms recursive formula and explicit formula. As students respond to questions or discuss possible solutions to problems, monitor their use of these terms to ensure their understanding and ability to use language correctly and precisely.

4. Use $a_1$ and $a_{n+1}$ to write a general expression for the common difference $d$.
   \[ d = a_{n+1} - a_n \]

5. Determine whether the numbers of carbon atoms in the first six alkanes
   \[ \{1, 2, 3, 4, 5, 6\} \]
   form an arithmetic sequence. Explain why or why not.
   Yes: The sequence is arithmetic because there is a common difference of 1 between each pair of consecutive terms.

MATH TIP

In a sequence, $a_{n+1}$ is the term that follows $a_n$.

Check Your Understanding

Determine whether each sequence is arithmetic. If the sequence is arithmetic, state the common difference.

6. 3, 8, 13, 18, 23, 
7. 1, 2, 4, 8, 16, 
8. Find the missing terms in the arithmetic sequence 19, 28, 
    37, 46, 55,

9. Write a formula for $a_{n+1}$ in Item 4.
   \[ a_{n+1} = a_n + d \]

10. What information is needed to find $a_{n+1}$ using this formula?
    The value of the common difference and the value of the previous term are needed.

Finding the value of $a_{n+1}$ in the formula you wrote in Item 9 requires knowing the value of the previous term. Such a formula is called a recursive formula, which is used to determine a term of a sequence using one or more of the preceding terms.

The terms in an arithmetic sequence can also be written as the sum of the first term and a multiple of the common difference. Such a formula is called an explicit formula because it can be used to calculate any term in the sequence as long as the first term is known.

11. Complete the blanks for the sequence \( \{4, 6, 8, 10, 12, 14, \ldots \} \) formed by the number of hydrogen atoms.

   \[
   \begin{align*}
   a_1 &= 4, & d &= 2 \\
   a_2 &= 4 + 1, & 2 &= 6 \\
   a_3 &= 4 + 2, & 2 &= 8 \\
   a_4 &= 4 + 3, & 2 &= 10 \\
   a_5 &= 4 + 4, & 2 &= 12 \\
   a_6 &= 4 + 5, & 2 &= 14 \\
   a_{10} &= 4 + 9, & 2 &= 22 \\
   \end{align*}
   \]
Lesson 19-1

Arithmetic Sequences

12. Write a general expression \( a_n \) in terms of \( n \) for finding the number of hydrogen atoms in an alkane molecule with \( n \) carbon atoms.
   \[ a_n = 4 + (n - 1) \cdot 2 \]

13. Use the expression you wrote in Item 12 to find the number of hydrogen atoms in decane, the alkane with 10 carbon atoms. Show your work.
   \[ a_{10} = 4 + (10 - 1) \cdot 2 = 4 + 9 \cdot 2 = 22 \]

14. Find the number of carbon atoms in a molecule of an alkane with 38 hydrogen atoms. \( n = 18 \)

15. Model with mathematics. Use \( a_n \), \( d \), and \( n \) to write an explicit formula for \( a_n \), the \( n \)th term of any arithmetic sequence.
   \[ a_n = a_1 + (n - 1) \cdot d \]

16. Use the formula from Item 15 to find the specified term in each arithmetic sequence.
   a. Find the 40th term when \( a_1 = 6 \) and \( d = 3 \). \( 123 \)
   b. Find the 30th term of the arithmetic sequence 37, 33, 29, 25, \ldots. \(-79 \)

Example A

Hope is sending invitations for a party. The cost of the invitations is $5.00, and postage for each is $0.45. Write an expression for the cost of mailing the invitations in terms of the number of invitations mailed. Then calculate the cost of mailing 16 invitations.

Step 1: Identify \( a_1 \) and \( d \).

The cost to mail the first invitation is equal to the cost of the invitations and the postage for that one invitation.
\[ a_1 = 5.00 + 0.45 = 5.45 \]
The postage per invitation is the common difference, \( d = 0.45 \).

Step 2: Use the information from Step 1 to write a general expression for \( a_n \).

If \( n \) equals the number of invitations mailed, then the expression for the cost of mailing \( n \) invitations is:
\[ a_n = a_1 + (n - 1)d \]
\[ a_n = 5.45 + (n - 1)(0.45) \]
\[ a_n = 5.45 + 0.45n - 0.45 \]
\[ a_n = 5.00 + 0.45n \]

Step 3: Use the general expression to evaluate \( a_{16} \).

The cost of mailing 16 invitations is found by solving for \( n = 16 \).
\[ a_{16} = 5.00 + 0.45(16) = 5.00 + 7.20 = 12.20 \]

Try These A

Write an expression for the \( n \)th term of the arithmetic sequence, and then find the term.

a. Find the 50th term when \( a_1 = 7 \) and \( d = -2 \).
\[ a_n = 7 + (n - 1) \cdot -2; \ a_{50} = -91 \]
b. Find the 28th term of the arithmetic sequence 3, 7, 11, 15, \ldots.
\[ a_n = 3 + (n - 1) \cdot 4; \ a_{28} = 111 \]
c. Which term in the arithmetic sequence 15, 18, 21, 24, \ldots is equal to 72?
\[ a_n = 15 + (n - 1) \cdot 3; \ n = 20 \]
### Activity 19 Continued

#### Check Your Understanding

Debrief students’ answers to these items to ensure that they understand how to use both explicit and recursive formulas to calculate the rth term of an arithmetic sequence. Ask students to think of situations in which one formula might be more useful than the other.

**Answers**

17. \(a_n = 4 + (n - 1) \cdot 2 = 4 + (2n - 2) = 2n + 2 = f(n)\)
18. \(-3.5\)
19. the 8th term
20. Shontelle can do this because \(a_n = a_{n-1} + d\). The results are equivalent; both formulas give the same result.

#### Assess

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

### Lesson 19-1 Practice

21. not arithmetic
22. a. \(-3\)
   
   b. \(a_n = 23 - 3n\)
   
   c. \(a_n = a_{n-1} - 3\)
23. a. \(d = 4\)
   
   b. \(a_n = 3 + 4(n - 1)\)
   
   c. \(a_n = a_{n-1} + 3\)
24. 13, 18, 23, 28, 33
25. \(a_1 = \frac{9}{8}\)

#### Connect to History

Item 21 is a famous sequence known as the Fibonacci sequence. Find out more about this interesting sequence. You can find its pattern in beehives, pinecones, and flowers.

#### Adapt

Check students’ answers to the Lesson Practice to ensure that they understand basic concepts related to arithmetic sequences and finding the rth term of such a sequence. Be sure students understand that \(d\) can be negative, as in Item 22. Students may substitute incorrectly into the formulas. Encourage them to begin each problem by writing the general formula and identifying each value in the formula. For example, for Item 23b, students can write \(a_n = a_1 + (n - 1)d\) followed by \(a_1 = 3, \ d = 4\). Correctly identifying and substituting these values will be important in upcoming lessons.
Learning Targets:
• Write a formula for the $n$th partial sum of an arithmetic series.
• Calculate partial sums of an arithmetic series.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Think-Pair-Share, Create Representations

A series is the sum of the terms in a sequence. The sum of the first $n$ terms of a series is the $n$th partial sum of the series and is denoted by $S_n$.

1. Consider the arithmetic sequence $\{4, 6, 8, 10, 12, 14, 16, 18\}$.
   a. Find $S_4$.
      $S_4 = 28$
   b. Find $S_5$.
      $S_5 = 40$
   c. Find $S_8$.
      $S_8 = 88$
   d. How does $a_1 + a_4$ compare to $a_2 + a_3$, $a_3 + a_4$, and $a_4 + a_5$?
      Each sum is 22.
   e. Make use of structure. Explain how to find $S_8$ using the value of $a_1 + a_4$.
      Since there are 4 pairs of numbers with the same sum, $S_8 = 4(a_1 + a_4) = 4(22) = 88$.

2. Consider the arithmetic series $1 + 2 + 3 + \ldots + 98 + 99 + 100$.
   a. How many terms are in this series?
      100
   b. If all the terms in this series are paired as shown below, how many pairs will there be?
      50
      \[
      \begin{array}{cccccccccc}
      1 & + & 2 & + & 3 & + & \ldots & + & 97 & + & 98 & + & 99 & + & 100
      \end{array}
      \]
   c. What is the sum of each pair of numbers?
      101
   d. Construct viable arguments. Find the sum of the series. Explain how you arrived at the sum.
      There are 50 pairs of numbers, each with a sum of 101, so the sum is $50(101) = 5050$.

CONNECT TO HISTORY
A story is often told that in the 1780s, a German schoolmaster decided to keep his students quiet by having them find the sum of the first 100 integers. One young pupil was able to name the sum immediately. This young man, Carl Friedrich Gauss, would become one of the world’s most famous mathematicians. He reportedly used the method in Item 2 to find the sum, using mental math.

Activity 19 • Arithmetic Sequences and Series
3. Consider the arithmetic series $a_1 + a_2 + a_3 + \ldots + a_{n-2} + a_{n-1} + a_n$.

a. Write an expression for the number of pairs of terms in this series.

b. Write a formula for $S_n$, the partial sum of the arithmetic series.

$$S_n = \frac{n}{2} (a_1 + a_n)$$

4. Use the formula from Item 3b to find each partial sum of the arithmetic sequence {4, 6, 8, 10, 12, 14, 16, 18}. Compare your results to your answers in Item 1.

a. $S_4$

$$S_4 = \frac{4}{2} (4 + 10) = 2(14) = 28$$

b. $S_5$

$$S_5 = \frac{5}{2} (4 + 12) = \frac{5}{2} (16) = 40$$

c. $S_8$

$$S_8 = \frac{8}{2} (4 + 18) = 4(22) = 88$$

ELL Support

Students may use the term series incorrectly because it sounds like a plural word and because it is sometimes used in everyday language to refer to a sequence. In fact, a common error is to use series interchangeably with sequence. Monitor students' use of series carefully to be sure they understand that the term refers to a single sum.
Lesson 19-2
Arithmetic Series

5. A second form of the formula for finding the partial sum of an arithmetic series is

\[ S_n = \frac{n}{2} [2a_1 + (n - 1)d] \]

Derive this formula, starting with the formula from Item 3b of this lesson and the \( n \)th term formula,

\[ a_n = a_1 + (n - 1)d \]

6. Use the formula \( S_n = \frac{n}{2} [2a_1 + (n - 1)d] \) to find the indicated partial sum of each arithmetic series. Show your work.
   a. \( 3 + 8 + 13 + \ldots \); \( S_{20} \)
   \[ S_{20} = \frac{20}{2} [2 \cdot 3 + (20 - 1) \cdot 5] = 10(6 + 95) = 1010 \]
   b. \( -2 - 4 - 6 - \ldots \); \( S_{18} \)
   \[ S_{18} = \frac{18}{2} [2 (-2) + (18 - 1)(-2)] = 9(-4 + 36 - 2) = 9(-38) = -342 \]

Example A
Find the partial sum \( S_{10} \) of the arithmetic series with \( a_1 = -3 \), \( d = 4 \).

Step 1: Find \( a_{10} \)
   The terms are \(-3, 1, 5, 9, \ldots \)
   \[ a_{10} = a_1 + (n - 1)d = -3 + (10 - 1)(4) = -3 + 36 = 33 \]

Step 2: Substitute for \( n, a_1, \) and \( a_{10} \) in the formula. Simplify.
   \[ S_{10} = \frac{10}{2} [2a_1 + (n - 1)d] = 5(-3 + 33) = 5(30) = 150 \]
   Or use the formula \( S_n = \frac{n}{2} [2a_1 + (n - 1)d] \):
   \[ S_{10} = \frac{10}{2} [2(-3) + (10 - 1)4] = 5[-6 + 36] = 150 \]

Try These A
Find the indicated sum of each arithmetic series. Show your work.
   a. Find \( S_5 \) for the arithmetic series with \( a_1 = 5 \) and \( a_5 = 40 \).
   \[ 180 \]
   b. \( 12 + 18 + 24 + \ldots \); \( S_{15} \)
   \[ 390 \]
   c. \( 30 + 20 + 10 + \ldots \); \( S_{25} \)
   \[ -2250 \]
Check Your Understanding

Debrief students’ answers to these items to ensure that they understand the concepts related to finding partial sums of arithmetic series. Ask students whether they could find the sum of all terms in an arithmetic sequence.

**Answers**

7. \( n = \frac{3}{2}, n = 6; a_1 = 12; a_6 = 37 \)

8. 12, 17, 22, 27, 32, 37; 12 + 17 + 22 + 27 + 32 + 37 = 147

9. Sample answer: Use \( S_n = \frac{n}{2}[2a_1 + (n - 1)d] \) when \( a_n \) is unknown.

**LESSON 19-2 PRACTICE**

10. \( S_{10} = 265 \)

11. \( S_{12} = 180 \)

12. \( S_{10} = 72 \)

13. 180 seats

**ADAPT**

Check students’ answers to the Lesson Practice to ensure that they understand basic concepts related to finding the \( n \)th partial sum of an arithmetic series. The goal of this lesson is for students to develop and use the formulas; watch for students who find sums by simply adding terms in the series. Insist that they show their work using the formulas.
Lesson 19-3
Sigma Notation

Learning Targets:
- Identify the index, lower and upper limits, and general term in sigma notation.
- Express the sum of a series using sigma notation.
- Find the sum of a series written in sigma notation.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Think-Pair-Share, Create Representations

In the Binomial Theorem activity in Unit 3, you were introduced to a shorthand notation called **sigma notation** (Σ). It is used to express the sum of a series.

The expression \( \sum_{n=1}^{4} (2n + 5) \) is read "the sum from \( n = 1 \) to \( n = 4 \) of \( 2n + 5 \)."

To expand the series to show the terms of the series, substitute 1, 2, 3, and 4 into the expression for the general term. To find the sum of the series, add the terms.

\[
\sum_{n=1}^{4} (2n + 5) = (2 \cdot 1 + 5) + (2 \cdot 2 + 5) + (2 \cdot 3 + 5) + (2 \cdot 4 + 5) = 7 + 9 + 11 + 13 = 40
\]

**Example A**

Evaluate \( \sum_{j=1}^{3} (2j - 3) \).

**Step 1:** The values of \( j \) are 1, 2, 3, 4, 5, and 6. Write a sum with six addends, one for each value of the variable.

\[
= [2(1) - 3] + [2(2) - 3] + [2(3) - 3] + [2(4) - 3] + [2(5) - 3] + [2(6) - 3]
\]

**Step 2:** Evaluate each expression.

\[
= -1 + 1 + 3 + 5 + 7 + 9
\]

**Step 3:** Simplify.

\[
= 24
\]

**Try These A**

- **a. Use appropriate tools strategically.** Write the terms in the sequence \( \sum_{n=1}^{5} (3n - 2) \). Then find the indicated sum.

\[
1 + 4 + 7 + 10 + 13 + 16 + 19 + 22 = 92
\]

- **b.** Write the sum of the first 10 terms of \( 80 + 75 + 70 + 65 + \ldots \) using sigma notation.

**Sample answer:** \( \sum_{n=1}^{10} (85 - 5n) \)

**MATH TIP**

The terms of a series written in sigma notation, substitute the value of the lower limit into the expression for the general term.

To find subsequent terms, substitute consecutive integers that follow the lower limit, stopping at the upper limit.

**MATH TIP**

To find the first term in a series written in sigma notation, substitute the value of the lower limit into the expression for the general term.

**Differentiating Instruction**

Support students who struggle with sigma notation by providing this additional practice.

Expand the series and find the sum.

1. \( \sum_{n=1}^{4} (3n + 1) \) \hspace{1cm} [50]
2. \( \sum_{n=1}^{4} (5 - 2n) \) \hspace{1cm} [−32]
3. \( \sum_{n=1}^{4} 2n \) \hspace{1cm} [110]

Then assign the following three items, in which the lower limit is not 1 and the upper limit is not the number of terms in the sequence.

Expand the series and find the sum.

4. \( \sum_{n=1}^{4} (3n + 1) \) \hspace{1cm} [3876]
5. \( \sum_{n=5}^{10} (5 - 2n) \) \hspace{1cm} [−6080]
6. \( \sum_{n=100}^{110} 2n \) \hspace{1cm} [991,100]

**ACTIVITY 19 Continued**

**Lesson 19-3**

**PLAN**

**Pacing:** 1 class period

**Chunking the Lesson**

Example A

Check Your Understanding

Lesson Practice

**TEACH**

**Bell-Ringer Activity**

Ask students to use one of the formulas for \( S_n \) from Lesson 19-2 to find the indicated partial sum of each arithmetic sequence.

1. \( 2, 4, 6, 8, \ldots ; S_4 \) [20]
2. \(-16, -6, 4, 14, 24, \ldots ; S_6 \) [54]
3. \( 5, 12, 19, 26, 31, \ldots ; S_8 \) [90]

Discuss the formulas students used to determine their answers.

**Example A Marking the Text, Simplify the Problem, Think-Pair-Share, Debriefing**

You might start by finding the value of \( 2j - 3 \) when \( j = 1 \) and \( j = 2 \). Now have students work in small groups to find the values of \( 2j - 3 \) for \( j = 3 \) through \( j = 6 \). Elicit the fact from students that they now must add all of these values because sigma notation represents their sum.

**TEACHER TO TEACHER**

Students were introduced to summation notation, or sigma notation, in Unit 3. Note that if the lower limit is 1, the upper limit is equal to the number of terms in the series. If the lower limit is an integer other than 1, say \( i \), and the upper limit is \( j \), then the number of terms in the series will be \( j - i + 1 \).
Check Your Understanding

Debrief students’ answers to these items to ensure that they know and understand how to calculate \( d \), \( a_n \), and \( S_n \). Ask them to explain the meaning of each term in their formulas.

**Answers**

1. \( d = a_{n+1} - a_n \)
2. \( a_n = a_1 + (n - 1) \cdot d \)
3. \( S_n = \frac{1}{2} (a_1 + a_n) \) or
   \[ S_n = \frac{n}{2} [2a_1 + (n - 1)d] \]

**Technology Tip**

Many graphing calculators and computer algebra systems can calculate sums like the ones in this lesson. Typically, Sum or Sequence functions return the sum when the general term, index of summation, and upper and lower limits are entered; consult your manual for specific instructions. You may wish to allow students to check their answers using technology until they gain confidence working with sigma notation.

For additional technology resources, visit SpringBoard Digital.

**ASSESS**

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

**LESSON 19-3 PRACTICE**

4. \[ \sum_{n=1}^{15} (3n - 1) \]
5. \[ \sum_{k=1}^{20} (2k + 1) \]
6. \[ \sum_{j=5}^{15} 3j \]
7. Identify the index, upper and lower limits, and general term of Item 4.
8. **Attend to precision.** Express the following sum using sigma notation: \( 3 + 7 + 11 + 15 + 19 + 23 + 27 \)

**ADAPT**

Check students’ answers to the Lesson Practice to ensure that they understand how to find a sum written in sigma notation. If students need more practice, find arithmetic sequences from the previous lessons in this activity and have students use sigma notation to write partial sums for these sequences. (Students may choose which partial sums, or you may wish to assign specific partial sums.) Students can then exchange papers and find each indicated sum.
Arithmetic Sequences and Series

Section 19-1

Determine whether or not each sequence is arithmetic. If the sequence is arithmetic, state the common difference.

1. a. 4, 5, 7, 10, . . .
   b. 5, 7, 9, 11, . . .
   c. 12, 9, 6, 3, . . .
2. a. 7, 7.5, 8, 8.5, . . .
   b. 6, 7, 8, 9, . . .
   c. −2, 4, −8, . . .
3. a. 4, 12, 20, 28, . . .
   b. 14, 18, 22, 26, . . .
   c. 45, 41, 37, 33, . . .
4. a. $a_{1} = 4$, $d = 5$; $a_{1}$
   b. $4, 18, 22, 26, . . .; a_{20}$
   c. $45, 41, 37, 33, . . .; a_{18}$
5. a. $a_{1} = 4$, $d = 5$; $a_{1}$
   b. $a_{20}$
   c. $a_{20}$
   d. $a_{20}$
6. a. $3, 6, 9, . . .$
   b. $= 32, 24, 16, . . .$
   c. $108, 96, 84, . . .$
   d. $-8, -4, 0, . . .$
7. If $a_{1} = 20$ and $a_{12} = 68$, find $a_{2}, a_{10},$ and $a_{p}$.
8. Find the first four terms of the sequence with $a_{1} = -2, d = 4$; $a_{12}$
   a. 4, 12, 10, 8, . . .
   b. 15, 19, 23, 27, . . .; $a_{10}$
   c. 46, 40, 34, 28, . . .; $a_{20}$

Activity 19 - continued

ACTIVITY PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 19-1

1. Determine whether or not each sequence is arithmetic. If the sequence is arithmetic, use the explicit formula to write a general expression for $a_{n}$. In terms of $n$.
   a. 4, 12, 20, 28, . . .
   b. 5, 10, 20, 40, . . .
   c. 4, 0, −4, −8, . . .
2. Determine whether or not each sequence is arithmetic. If the sequence is arithmetic, use the explicit formula to write a general expression for $a_{n}$, in terms of $a_{n-1}$.
   a. 7, 7.5, 8, 8.5, . . .
   b. 6, 7, 8, 9, . . .
   c. −2, 4, −8, . . .
3. Find the indicated term of each arithmetic sequence.
   a. $a_{3} = 4, d = 5$; $a_{13}$
   b. $14, 18, 22, 26, . . .; a_{20}$
   c. $45, 41, 37, 33, . . .; a_{18}$
4. Find the sequence for which $a_{n}$ is arithmetic. If the sequence is arithmetic, state the common difference.
   a. $a_{1} = 4, d = 5$; $a_{13}$
   b. $14, 18, 22, 26, . . .; a_{20}$
   c. $45, 41, 37, 33, . . .; a_{18}$
5. Find the first four terms of the sequence with $a_{1} = 2, a_{2} = 8, a_{3} = 14$.
   a. $a_{4}$
   b. $a_{10}$
   c. $a_{20} = -68$

Lesson 19-2

Find the indicated partial sum of each arithmetic series.

12. Find the indicated partial sum of each arithmetic series.
   a. $a_{1} = 4, d = 5$; $S_{10}$
   b. $14 + 18 + 22 + 26 + . . .; S_{12}$
   c. $45 + 41 + 37 + 33 + . . .; S_{18}$
13. Find the indicated partial sum of each arithmetic series.
   a. $1 + 3 + 5 + . . .; S_{6}$
   b. $1 + 3 + 5 + . . .; S_{10}$
   c. $1 + 3 + 5 + . . .; S_{12}$
   d. $S_{n}$
14. Find the indicated partial sum of the arithmetic series.
   a. $0 + (x + 2) + (2x + 4) + (3x + 6) + . . .; S_{10}$
   b. $10x + 20$
   c. $45x + 90$
   d. $5x + 110$
15. Two companies offer you a job. Company A offers you a $40,000 first-year salary with an annual raise of $1500. Company B offers you a $38,500 first-year salary with an annual raise of $2000.
   a. What would be your total earnings with Company A as you begin your sixth year?
   b. What would be your total earnings with Company B as you begin your sixth year?
   c. What would be your total earnings with Company A after 5 years?
   d. What would be your total earnings with Company B after 5 years?
ACTIVITY 19

16. If $S_{12} = 744$ and $a_1 = 40$, find $d$.
17. In an arithmetic series, $a_1 = 47$ and $a_2 = -13$, find $d$ and $S_n$.
18. In an arithmetic series, $a_n = 9.44$ and $d = 0.4$, find $a_1$ and $S_n$.
19. The first prize in a contest is $500, the second prize is $450, the third prize is $400, and so on.
   a. How many prizes will be awarded if the last prize is $100?
   b. How much money will be given out as prize money?
20. Find the sum of the first 150 natural numbers.
   A. 1339
   B. 1648
   C. 1930
   D. 2060
21. Find the sum of the first 150 natural numbers.
22. A store puts boxes of canned goods into a stacked display. There are 20 boxes in the bottom layer. Each layer has two fewer boxes than the layer below it. There are five layers of boxes. How many boxes are in the display? Explain your answer.
23. Find the indicated partial sum of each arithmetic series.
   a. $\sum_{j=1}^{10} (5 - 6j)$
   b. $\sum_{j=1}^{20} 5j$
   c. $\sum_{j=1}^{15} (5 - j)$
24. Does $\sum_{j=1}^{10} (2j + 1) = 120$?
   a. If yes, $\sum_{j=1}^{20} (2j + 1) = 35$; $\sum_{j=1}^{10} (2j + 1) = 85$; $\sum_{j=1}^{20} (2j + 1) = 35 + 85$
   b. If yes, $\sum_{j=1}^{9} (j - 7) = -3$; $\sum_{j=1}^{10} (j - 7) = -18$; $\sum_{j=1}^{15} (j - 7) = -15$; $-3 = -18 - (-15)$
25. B
26. B
27. a. $\sum_{j=1}^{3} (j + 3) = 39$
   b. $\sum_{j=1}^{15} (j - 12) = 3$
   c. $\sum_{j=1}^{8} (4j) = 144$
28. $\sum_{j=1}^{8} (-3j + 29) = 55$ and $\sum_{j=1}^{8} (-3j + 29) = 61$; $\sum_{j=1}^{8} (-3j + 29)$ is greater.
29. D
30. $\sum_{j=1}^{8} \left( \frac{j - \pi}{2} \right) = \frac{15\pi}{2}$
31. Sample answer: Because there is a constant difference between sequential terms of a sequence, terms can be paired to represent a constant sum. Therefore, the partial sum is the product of the number of pairs times the sum.

ADDITIONAL PRACTICE
If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

Arithmetic Sequences and Series

ACTIVITY 19

16. Which statement is true for the partial sum $\sum_{j=1}^{4} (4j + 3)$?
   A. For $n = 5$, the sum is 35.
   B. For $n = 7$, the sum is 133.
   C. For $n = 10$, the sum is 230.
   D. For $n = 12$, the sum is 408.
27. Evaluate.
   a. $\sum_{j=1}^{6} (j + 3)$
   b. $\sum_{j=1}^{10} (j - 12)$
   c. $\sum_{j=1}^{8} (4j)$
28. Which is greater: $\sum_{j=1}^{4} (-3j + 29)$ or $\sum_{j=1}^{4} (-3j + 29)$?
29. Which expression is the sum of the series $7 + 10 + 13 + \ldots + 25$?
   A. $\sum_{j=1}^{4} 4 + 3j$
   B. $\sum_{j=1}^{4} (4 - 3j)$
   C. $\sum_{j=1}^{4} (3 + 4j)$
   D. $\sum_{j=1}^{4} (4 + 3j)$
30. Evaluate $\sum_{j=1}^{8} \left( \frac{j - \pi}{2} \right)$.

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

31. How does the common difference of an arithmetic sequence relate to finding the partial sum of an arithmetic series?
Learning Targets:
- Determine whether a given sequence is geometric.
- Find the common ratio of a geometric sequence.
- Write an expression for a geometric sequence, and calculate the nth term.

**SUGGESTED LEARNING STRATEGIES:** Summarizing, Paraphrasing, Create Representations

Meredith is designing a mural for an outside wall of a warehouse that is being converted into the Taylor Modern Art Museum. The mural is 32 feet wide by 31 feet high. The design consists of squares in five different sizes that are painted black or white as shown below.

<table>
<thead>
<tr>
<th>Square #</th>
<th>Side of Square (ft)</th>
<th>Number of Squares</th>
<th>Area of Square (ft²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>2</td>
<td>256</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>4</td>
<td>64</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>8</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>32</td>
<td>1</td>
</tr>
</tbody>
</table>

1. Let Square 1 be the largest size and Square 5 be the smallest size. For each size, record the length of the side, the number of squares of that size in the design, and the area of the square.

**Lesson 20-1**

**ACTIVITY 20**

**Guided**

**Activity Standards Focus**

In Activity 20, students learn about geometric sequences and series. First they learn to identify and define a geometric sequence, including identifying the common ratio. Then they will examine and find sums of finite and infinite geometric series. Students will use information they learned in the previous activity about writing sequences and series in explicit and recursive forms.

**Common Core State Standards for Activity 20**

- HSA-SSE.B.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.*
- HSF-BF.A.1 Write a function that describes a relationship between two quantities.*
- HSF-BF.A.1a Determine an explicit expression, a recursive process, or steps for calculation from a context.
- HSF-BF.A.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.*
2. Refer to the table in Item 1.
   a. Describe any patterns that you notice in the table.
      Sample answer: As the square number increases by 1, the side length of a square is divided by 2. The number of squares doubles each time the square number increases by 1. The areas are perfect squares. Each product of the side of a square and the number of squares is 32.
   b. Each column of numbers forms a sequence of numbers. List the four sequences that you see in the columns of the table.
      1, 2, 3, 4, 5; 16, 8, 4, 2, 1; 2, 4, 8, 16, 32; 256, 64, 16, 4, 1
   c. Are any of those sequences arithmetic? Why or why not?
      The only sequence that is arithmetic is 1, 2, 3, 4, 5, because it is the only sequence with a common difference. The common difference is 1.

3. Consider the sequences in Item 2b.
   a. List those sequences that are geometric.
      Side of square: 16, 8, 4, 2, 1
      Number of squares: 2, 4, 8, 16, 32
      Area of square: 256, 64, 16, 4, 1
   b. State the common ratio for each geometric sequence.
      Side of square: \( r = \frac{1}{2} \)
      Number of squares: \( r = 2 \)
      Area of square: \( r = \frac{1}{4} \)
Lesson 20-1
Geometric Sequences

4. Use \(a_n\) and \(a_{n-1}\) to write a general expression for the common ratio \(r\).
\[ r = \frac{a_n}{a_{n-1}} \]

5. Consider the sequences in the columns of the table in Item 1 that are labeled Square # and Side of Square.
   a. Plot the Square # sequence by plotting the ordered pairs (term number, square number).
   b. Using another color or symbol, plot the Side of Square sequence by plotting the ordered pairs (term number, side of square).
   c. Is either sequence a linear function? Explain why or why not.
      The Square # plot is linear because there is a constant rate of change. Each time the term number increases by 1, the Square # increases by 1. The Side of Square sequence does not increase by a constant rate of change. Each time the term number increases by 1, the Side of Square decreases by an increasingly smaller amount.

Check Your Understanding

6. Determine whether each sequence is arithmetic, geometric, or neither.
   a. 3, 6, 12, 24, . . .
   b. 5, 10, 20, 40, . . .
   c. 4, 8, 12, 16, . . .
   d. 2, 4, 8, 16, . . .

7. Use \(a_{n+1}\) and \(a_{n-1}\) to write an expression for the common ratio \(r\).

8. Describe the graph of the first 5 terms of a geometric sequence with the first term 2 and the common ratio equal to 1.

9. Reason abstractly. Use the expression from Item 4 to write a recursive formula for the term \(a_n\) and describe what the formula means.
   \[ a_n = a_{n-1}r \]
   We can find a term in a geometric sequence by knowing the previous term and the common ratio.
The terms in a geometric sequence also can be written as the product of the first term and a power of the common ratio.

10. For the geometric sequence \(\{4, 8, 16, 32, 64, \ldots\}\), identify \(a_1\) and \(r\). Then fill in the missing exponents and blanks.

- \(a_1 = \) 4
- \(r = \) \(2^n\) for each term

\[
\begin{align*}
a_1 &= 4 \\
a_2 &= 4 \cdot 2^1 = 8 \\
a_3 &= 4 \cdot 2^2 = 16 \\
a_4 &= 4 \cdot 2^3 = 32 \\
a_5 &= 4 \cdot 2^4 = 64 \\
a_6 &= 4 \cdot 2^5 = 128 \\
a_{10} &= 4 \cdot 2^9 = 32 \text{,768}
\end{align*}
\]

11. Use \(a_1\), \(r\), and \(n\) to write an explicit formula for the \(n\)th term of any geometric sequence.

\[a_n = a_1 \cdot r^{n-1}\]

12. Use the formula from Item 11 to find the indicated term in each geometric sequence.

- a. \(1, 2, 4, 8, 16, \ldots\); \(a_{16}\) \(= 32\text{,}768\)

- b. \(4096, 1024, 256, 64, \ldots\); \(a_9\) \(= \frac{1}{16}\) or 0.0625
Check Your Understanding

13. a. Complete the table for the terms in the sequence with $a_1 = 3; r = 2$.

<table>
<thead>
<tr>
<th>Term</th>
<th>Recursive</th>
<th>Explicit</th>
<th>Value of Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>3</td>
<td>$3 \cdot 2^{1-1} = 3$</td>
<td>3</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$3 \cdot 2$</td>
<td>$3 \cdot 2^{2-1} = 3 \cdot 2$</td>
<td>6</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$(3 \cdot 2) \cdot 2$</td>
<td>$3 \cdot 2^{3-1} = 3 \cdot 2^2$</td>
<td>12</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$((3 \cdot 2) \cdot 2) \cdot 2$</td>
<td>$3 \cdot 2^{4-1} = 3 \cdot 2^3$</td>
<td>24</td>
</tr>
<tr>
<td>$a_5$</td>
<td>$(((3 \cdot 2) \cdot 2) \cdot 2) \cdot 2$</td>
<td>$3 \cdot 2^{5-1} = 3 \cdot 2^4$</td>
<td>48</td>
</tr>
</tbody>
</table>

b. What does the product $(3 \cdot 2)$ represent in the recursive expression for $a_3$?

c. Express regularity in repeated reasoning. Compare the recursive and explicit expressions for each term. What do you notice?

14. Write a formula that will produce the sequence that appears on the calculator screen below.

15. Determine whether each sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, state the common difference, and if it is geometric, state the common ratio.

a. 3, 5, 7, 9, 11, . . .

b. 5, 15, 45, 135, . . .

c. 6, $-4 \frac{8}{3}$, $-16 \frac{16}{9}$, . . .

d. 1, 2, 4, 7, 11, . . .

16. Find the indicated term of each geometric sequence.

a. $a_3 = -2, r = 3; a_4$

b. $a_4 = 1024, r = -\frac{1}{2}; a_{12}$

17. Attend to precision. Given the data in the table below, write both a recursive formula and an explicit formula for $a_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n$</td>
<td>0.25</td>
<td>0.75</td>
<td>2.25</td>
<td>6.75</td>
</tr>
</tbody>
</table>
ACTIVITY 20 Continued
Lesson 20-2

PLAN

Materials
- paper squares
- scissors

Pacing: 1 class period

Chucking the Lesson
#1 Example A
Check Your Understanding #4a #4b–d #5–6
Check Your Understanding Lesson Practice

TEACH

Bell-Ringer Activity
Ask students to find the eighth term in each geometric sequence.
1. 4, 16, 64, 256, … [65,536]
2. 1, –3, 9, –27, … [–16,384]
3. 4, 2, 1, 0.5, … \left[ \frac{1}{2} \right]

1 Close Reading, Think-Pair-Share
Look for groups of students who are able to follow the steps to derive the formula successfully, and have them share their work with the class.

TEACHER TO TEACHER

Students focus on a formula for determining the sum of a finite geometric series:
$$S_n = a_1 \left( \frac{1-r^n}{1-r} \right), \ r \neq 1.$$ They follow the steps given to derive the formula and then apply it. A common student error when applying the formula is to substitute the common ratio with the other term in the sequence.

Example A Think-Pair-Share, Debriefing
This example verifies that the formula results in the desired sum.

ELL Support
Students have encountered a lot of math terms that sound very similar: arithmetic series and arithmetic sequence, geometric series and geometric sequence, common difference and common ratio, etc. Provide linguistic support through translations of these terms and other language. As appropriate, remind students to refer to the English-Spanish glossary to aid their comprehension. Finally, monitor classroom discussions for understanding and correct language use.

MATH TERMS

A finite series is the sum of a finite sequence and has a specific number of terms.
An infinite series is the sum of an infinite sequence and has an infinite number of terms. You will work with infinite series later in this Lesson.

MATH TIP

When writing out a sequence, separate the terms with commas. A series is written out as an expression and the terms are separated by addition symbols. If a series has negative terms, then the series may be written with subtraction symbols.

Learning Targets:
- Derive the formula for the sum of a finite geometric series.
- Calculate the partial sums of a geometric series.

SUGGESTED LEARNING STRATEGIES:
Close Reading, Vocabulary Organizer, Think-Pair-Share, Create Representations

The sum of the terms of a geometric sequence is a geometric series. The sum of a finite geometric series where \( r \neq 1 \) is given by these formulas:
$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1}$$
$$S_n = a_1 \left( \frac{1-r^n}{1-r} \right)$$

1. To derive the formula, Step 1 requires multiplying the equation of the sum by \( -r \). Follow the remaining steps on the left to complete the derivation of the sum formula.

Step 1
$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1}$$
$$-rS_n = -a_1r - a_1r^2 - a_1r^3 - \ldots - a_1r^{n-1} - a_1r^n$$

Step 2
Combine terms on each side of the equation (most terms will cancel out).
$$S_n - rS_n = a_1 - a_1r^n\quad\text{Step 3 Factor out }S_n\text{ on the left side of the equation and factor out }a_1\text{ on the right.}$$
$$S_n(1 - r) = a_1(1 - r^n)\quad\text{Step 4 Solve for }S_n.$$ 
$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$ 

Example A
Find the total of the Area of Square column in the table in Item 1 from the last lesson. Then use the formula developed in Item 1 of this lesson to find the total area and show that the result is the same.

Step 1: Add the areas of each square from the table.
$$256 + 64 + 16 + 4 + 1 = 341$$

<table>
<thead>
<tr>
<th>Area #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area</td>
<td>256</td>
<td>64</td>
<td>16</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Step 2: Find the common ratio.
$$\frac{64}{256} = 0.25, \frac{16}{64} = 0.25, \frac{4}{16} = 0.25; r = 0.25$$

Step 3: Substitute \( n = 5 \), \( a_1 = 256 \), and \( r = 0.25 \) into the formula for \( S_n \).
$$S_n = a_1 \left( \frac{1-r^n}{1-r} \right), S_5 = 256 \left( \frac{1-0.25^5}{1-0.25} \right)$$

Step 4: Evaluate \( S_5 \).
$$S_5 = 256 \left( \frac{1-0.25^5}{1-0.25} \right) = 341$$
Lesson 20-2
Geometric Series

Try These A
Find the indicated sum of each geometric series. Show your work.

a. Find $S_3$ for the geometric series with $a_1 = 5$ and $r = 2$. $155$

b. $256 + 64 + 16 + 4 + \ldots; S_6$ $341.25$

c. $\sum_{n=1}^{5} 2 \cdot 3^{n-1}$ $59,048$

Check Your Understanding

2. **Reason quantitatively.** How do you determine if the common ratio in a series is negative?
3. Find the sum of the series $2 + 8 + 32 + 128 + 512$ using sigma notation.

Recall that the sum of the first $n$ terms of a series is a *partial sum*. For some geometric series, the partial sums $S_1, S_2, S_3, \ldots$ form a sequence with terms that approach a limiting value. The limiting value is called the sum of the *infinite geometric series*.

To understand the concept of an infinite sum of a geometric series, follow these steps:
- Start with a square piece of paper, and let it represent one whole unit.
- Cut the paper in half, place one piece of the paper on your desk, and keep the other piece of paper in your hand. The paper on your desk represents the first partial sum of the series, $S_1 = \frac{1}{2}$.
- Cut the paper in half again, adding one of the pieces to the paper on your desk and keeping the other piece in your hand. The paper on your desk now represents the second partial sum.
- Repeat this process as many times as you are able.

4. **Use appropriate tools strategically.** Each time you add a piece of paper to your desk, the paper represents the next term in the geometric series.
   a. As you continue the process of placing half of the remaining paper on your desk, what happens to the amount of paper on your desktop? The amount of paper on the desk gets closer to 1, the amount represented by the original square.

**ACTIVITY 20**

**MATH TIP**
Recall that sigma notation is a shorthand notation for a series. For example:

$$\sum_{n=1}^{3} 8 \cdot 2^{n-1} = 8(2^{1-1}) + 8(2^{2-1}) + 8(2^{3-1}) = 8 \cdot 1 + 8 \cdot 2 + 8 \cdot 4 = 8 + 16 + 32 = 56$$

**MATH TIP**
If the terms in the sequence $a_1, a_2, a_3, \ldots$ get close to some constant as $n$ gets very large, the constant is the limiting value of the sequence. For example, in the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots$ the terms get closer to a limiting value of 0 as $n$ gets larger.

**Check Your Understanding**

2. The terms will have signs alternating between positive and negative.
3. $\sum_{j=1}^{\infty} 2(4)^{j-1}$

**Teacher to Teacher**

The final part of this lesson deals with infinite sums. The concept of the existence of an infinite sum is difficult for many students. A geometric series can have a finite or infinite number of terms. The sum $S_n$ of the first $n$ terms of an infinite geometric series is called the $n$th partial sum of the series.

If the sequence of partial sums $S_1, S_2, S_3, \ldots$ approaches some specific number $S$, then the geometric series is said to have that number $S$ as its sum. The paper-cutting activity provides a concrete example of partial sums of an infinite series approaching a specific value.

**Developing Math Language**
Discuss with students that a *partial sum of a series* is a type of *finite series*. This will help students to see how some of the new vocabulary terms are related. Reinforce students’ acquisition of all vocabulary through regular reference to words in the text as well as words you and students place on the classroom Word Wall.

4. **Use Manipulatives** Students will realize that the longer they continue the process of adding half the remaining paper to their desks, the closer the amount of paper on their desks gets to a complete square, which represents one whole.

**MINI-LESSON:** Partial Sums of Geometric Series

If students need additional help with finding partial sums of geometric series, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.
ACTIVITY 20 Continued

4b–d Create Representations, Debriefing  Students will also see the limiting value of 1 by looking at the problem numerically and graphically.

To illustrate convergence and divergence, have students graph the partial sums for each infinite geometric series below.

\[ S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \]
\[ S_n = \frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \ldots \]

The graphs show that the first series converges and the second diverges.

4. b. Fill in the blanks to complete the partial sums for the infinite geometric series represented by the pieces of paper on your desk.

\[ S_1 = \frac{1}{2} \]
\[ S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \]
\[ S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \]
\[ S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{15}{16} \]
\[ S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32} \]
\[ S_6 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{63}{64} \]

4c. Plot the first six partial sums.

4d. Do the partial sums appear to be approaching a limiting value? If so, what does the value appear to be?

Yes, the limiting value appears to be one.
5. Consider the geometric series \(2 + 4 + 8 + 16 + 32 + \ldots\).
   a. List the first five partial sums for this series.
      \[ S_1 = 2; S_2 = 6; S_3 = 14; S_4 = 30; S_5 = 62 \]
   b. Do these partial sums appear to have a limiting value?
      No; there does not appear to be a limiting value.
   c. Does there appear to be a sum of the infinite series? If so, what does the sum appear to be? If not, why not?
      No; there does not appear to be an infinite sum. Since the terms that are being added in the partial sums are growing larger and larger, there will not be a limiting value on the sums.

6. Consider the geometric series \(3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} - \ldots\).
   a. List the first seven partial sums for this series.
      \[ S_1 = 3; S_2 = 2; S_3 = \frac{7}{3} \approx 2.333; S_4 = \frac{20}{9} \approx 2.222; S_5 = \frac{61}{27} \approx 2.259; S_6 = \frac{182}{81} \approx 2.247; S_7 = \frac{547}{243} \approx 2.251 \]
   b. Do these partial sums appear to have a limiting value?
      Yes; about 2.25.
   c. Does there appear to be a sum of the infinite series? If so, what does the sum appear to be? If not, why not?
      Yes; the sum appears to be 2.25.

**WRITING MATH**

You can write the sum of an infinite series by using summation or sigma, notation and using an infinity symbol for the upper limit. For example,
\[
\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} = 3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \ldots
\]

**ACTIVITY 20 Continued**

5–6 Look for a Pattern, Quickwrite, Debriefing These items provide one example in which the infinite sum of a geometric series does not exist (Item 5) and one in which the sum does exist (Item 6).

**TEACHER TO TEACHER**

The study of infinite geometric series lays a foundation for the study of infinite series in calculus. In a calculus course, students will use the nth-term test for divergence, which states that if a sequence \(\{a_n\}\) does not converge to 0, then the series \(\sum a_n\) diverges. In other words, in Item 5, since the terms 2, 4, 8, 16, 32, \ldots do not approach 0, the infinite sum \(2 + 4 + 8 + 16 + 32 + \ldots\) diverges, or does not exist.

In general, the converse of this test is not true, and thus it can never be used to prove convergence. In Item 6, you could not use the test to say that because the terms \(-\frac{1}{3}, -\frac{1}{9}, -\frac{1}{27}, \ldots\) approach 0, the sum \(3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \ldots\) must exist.
Check Your Understanding

Debrief students’ answers to these items to ensure that they can write partial sums of an infinite series. Ask students to explain the relationship between the terms of a sequence and the partial sums.

Answers

7. 1; 3; 7; 15; 31; 63; 127; 255; no limit; no infinite sum
8. \[\frac{2}{3}, \frac{8}{15}, \frac{26}{45}, \frac{80}{135}, \frac{242}{270}, \frac{728}{728}\]

LESSON 20-2 PRACTICE

9. \(S_7 = 547\)
10. \(S_9 = 781.2496\)
11. The sum is 0 when \(n\) is even.
12. The sum is \(-1\) when \(n\) is odd.
13. The sums oscillate between 0 and \(-1\).
14. When \(r\) is greater than or equal to 1, or less than or equal to \(-1\), the partial sums do not appear to have a limiting value.

Check students’ answers to the Lesson Practice to ensure that they understand how to find a particular partial sum without finding all the previous sums. If students make errors using the formula

\[S_n = a \left(\frac{1-r^n}{1-r}\right)\]

be sure they are using the proper order of operations. For example, they must find the value of \(r^n\) before subtracting in the numerator. If they are using a calculator, they must be sure that the calculator subtracts in the numerator and in the denominator before dividing or multiplying; this will require either using parentheses or performing the calculation in several steps.
Lesson 20-3
Convergence of Series

Learning Targets:
• Determine if an infinite geometric sum converges.
• Find the sum of a convergent geometric series.

SUGGESTED LEARNING STRATEGIES: Create Representations,
Look for a Pattern, Quickwrite

Recall the formula for the sum of a finite series \( S_n = \frac{a_1(1-r^n)}{1-r} \). To find the sum of an infinite series, find the value that \( S_n \) gets close to as \( n \) gets very large. For any infinite geometric series where \(-1 < r < 1\), as \( n \) gets very large, \( r^n \) gets close to 0.

\[
S_n = \frac{a_1(1-r^n)}{1-r} \approx \frac{a_1(1-0)}{1-r} \approx \frac{a_1}{1-r}
\]

An infinite geometric series \( \sum_{n=0}^{\infty} a_n r^n \) converges to the sum \( S = \frac{a_1}{1-r} \) if and only if \( |r| < 1 \) or \(-1 < r < 1\). If \( |r| \geq 1 \), the infinite sum does not exist.

1. Consider the three series from Items 4–6 of the previous lesson. Decide whether the formula for the sum of an infinite geometric series applies. If so, use it to find the sum. Compare the results to your previous answers.
   a. \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \) \( S = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \); the results are the same.
   b. \( 2 + 4 + 8 + 16 + 32 + \ldots \) Does not apply since \( r \geq 1 \).
   c. \( 3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} + \ldots \) \( S = \frac{3}{1-\left(-\frac{1}{3}\right)} = \frac{9}{4} \approx 2.25 \); the results are the same.

Check Your Understanding

Find the infinite sum if it exists or tell why it does not exist. Show your work.

2. \( 64 + 16 + 4 + 1 + \ldots \)
   \( S = \frac{256}{3} = 85.\overline{3} \)

3. does not exist because \( r = \frac{5}{4} \geq 1 \)

4. \( \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} \)

Answers

2. \( S = \frac{256}{3} = 85.\overline{3} \)

3. does not exist because \( r = \frac{5}{4} \geq 1 \)

4. \( S = 3 + \frac{6}{5} + \frac{12}{25} + \ldots = \frac{3}{5} \)
ACTIVITY 20  Continued

5–6 Look for a Pattern, Quickwrite, Think-Pair-Share, Debriefing Item 5 is included in case students question whether or not infinite arithmetic series have infinite sums. Item 6 gives students the opportunity to reflect on all the formulas derived in the activity. After sharing answers with the entire group to make sure all responses are correct, students can record the information in their math notebooks.

Check Your Understanding
Debrief students’ answers to these items to ensure that they understand and can verify that the series is geometric. These items make a connection between infinite geometric series and repeating decimals, which students have prior experience with. After Item 8, ask students to find the decimal value of \( \frac{2}{9} \) and compare it to the series.

Answers
7. \( r = 0.10 \)
8. yes; \( S = \frac{a_1}{1 - r} = \frac{0.2}{1 - 0.1} = \frac{0.2}{0.9} = \frac{2}{9} \)
9. \( \frac{5}{9}, \frac{5}{9} = \frac{a_1}{1 - r} = \frac{0.5}{1 - 0.1} = \frac{5}{9} \)

ASSESS
Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 20–3 PRACTICE
10. \( S = 12 \)
11. \( S = 2187 \)
12. does not exist because \( r = \frac{4}{3} \geq 1 \)

ADAPT
Check students’ answers to the Lesson Practice to ensure that they understand how to determine whether an infinite sum exists for a geometric series. Watch for students who attempt to find an infinite sum without first checking whether it exists. Remind them that infinite sums may not exist, and demonstrate using examples from this activity.

ACTIVITY 20  continued

5. Consider the arithmetic series 2 + 5 + 8 + 11 + \ldots .
   a. Find the first four partial sums of the series.
      \( S_1 = 2; S_2 = 7; S_3 = 15; S_4 = 26 \)
   b. Do these partial sums appear to have a limiting value? No; there does not appear to be a limiting value.
   c. Does the arithmetic series appear to have an infinite sum? Explain. No; there does not appear to be an infinite sum. Since the terms that are being added in the partial sums are growing larger and larger, there will not be a limiting value on the sums.

6. Summarize the following formulas for a geometric series.
   \[
   \text{common ratio} \quad r = \frac{a_n}{a_{n-1}} \\
   \text{n}th \text{ term} \quad a_n = a_1 \cdot r^{n-1} \\
   \text{Sum of first n terms} \quad S_n = a_1 \left(1 - r^n\right) \left(1 - r\right) \\
   \text{Infinite sum} \quad S = \frac{a_1}{1 - r} \quad \text{if} \quad -1 < r < 1
   \]

Check Your Understanding
Consider the series 0.2 + 0.02 + 0.002 + \ldots .
7. Find the common ratio between the terms of the series.
8. Does this series have an infinite sum? If yes, use the formula to find the sum.
9. Construct viable arguments. Make a conjecture about the infinite sum 0.5 + 0.05 + 0.005 + \ldots . Then verify your conjecture with the formula.

LESSON 20–3 PRACTICE
Find the infinite sum if it exists, or tell why it does not exist.
10. \( 18 - 9 + \frac{9}{2} - \frac{9}{4} + \ldots \)
11. \( 729 + 486 + 324 + 216 + \ldots \)
12. \( 81 + 108 + 144 + 192 + \ldots \)
13. \( -33 - 66 - 99 - 132 \ldots \)
14. Reason quantitatively. At the beginning of the lesson it is stated that “for any infinite geometric series where \(-1 < r < 1\), as \(n\) gets very large, \(r^n\) gets close to 0.” Justify this statement with an example, using a negative value for \(r\).
Geometric Sequences and Series
Squares with Patterns

ACTIVITY 20 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 20-1
1. Write arithmetic, geometric, or neither for each sequence. If arithmetic, state the common difference. If geometric, state the common ratio.
   a. 4, 12, 36, 108, 324, . . .
   b. 1, 2, 6, 24, 120, . . .
   c. 4, 9, 14, 19, 24, . . .
   d. 35, −30, 25, −20, 15, . . .
2. Find the indicated term of each geometric series.
   a. \(a_1 = 1, r = −3; a_{10}\)
   b. \(a_1 = 3072, r = \frac{1}{4}; a_3\)
3. If \(a_n\) is a geometric sequence, express the quotient of \(a_{n+1}\) in terms of \(r\).
4. The first three terms of a geometric series are \(\frac{2}{81}, \frac{1}{27}, \frac{1}{9}; \ldots\) What is \(a_3\)?
   A. \(\frac{3}{81}\)
   B. 3
   C. \(\frac{364}{81}\)
   D. 9
5. Determine the first three terms of a geometric sequence with a common ratio of 2 and defined as follows:
   \(x − 1, x + 6, 3x + 4\)
6. Determine whether each sequence is geometric. If it is a geometric sequence, state the common ratio.
   a. \(x, x^2, x^4, \ldots\)
   b. \((x + 3), (x + 3)^2, (x + 3)^3, \ldots\)
   c. \(3^2, 3^3, 3^4, \ldots\)
   d. \(x^2, (2x)^2, (3x)^2, \ldots\)
7. If \(a_1 = \frac{2}{32}\) and \(a_2 = \frac{81}{512}\), find \(a_1\) and \(r\).
8. The \(S\) in the expression \(a_n = 4(5)^{n−1}\) represents which part of the expression?
   A. \(n\)
   B. \(a_n\)
   C. \(r\)
   D. \(S_n\)
9. A ball is dropped from a height of 24 feet. The ball bounces to 85% of its previous height with each bounce. Write an expression and solve to find how high (to the nearest tenth of a foot) the ball bounces on the sixth bounce.
10. Write the recursive formula for each sequence.
   a. 4, 2, 1, 0.5, . . .
   b. 2, 6, 18, 54, 120, . . .
   c. \(\frac{1}{5}, \frac{4}{25}, \frac{16}{125}, \ldots\)
   d. −45, 5, −\(\frac{5}{9}\), . . .
11. Write the explicit formula for each sequence.
   a. 4, 2, 1, 0.5, . . .
   b. 2, 6, 18, 54, 120, . . .
   c. \(\frac{1}{5}, \frac{4}{25}, \frac{16}{125}, \ldots\)
   d. −45, 5, −\(\frac{5}{9}\), . . .

Lesson 20-2
12. Find the indicated partial sum of each geometric series.
   a. \(5 + 2 + \frac{4}{5} + \frac{8}{25} + \ldots; S_7\)
   b. \(\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \ldots; S_5\)
13. For the geometric series \(2.9 + 3.77 + 4.90 + 6.37 + \ldots\), do the following:
   a. Find \(S_n\) (to the nearest hundredth).
   b. How many more terms have to be added in order for the sum to be greater than 200?
14. George and Martha had two children by 1776, and each child had two children. If this pattern continued to the 12th generation, how many descendants do George and Martha have?
15. A finite geometric series is defined as \(0.6 + 0.84 + 1.12 + 1.65 + \ldots + 17.36\). How many terms are in the series?
   A. \(n = 5\)
   B. \(n = 8\)
   C. \(n = 10\)
   D. \(n = 11\)
16. Evaluate \(\sum_{j=1}^{6} (3j)^{\frac{1}{2}}\).
ACTIVITY 20

17. During a 10-week summer promotion, a baseball team is letting all spectators enter their names in a weekly drawing each time they purchase a game ticket. Once a name is in the drawing, it remains in the drawing unless it is chosen as a winner. Since the number of names in the drawing increases each week, so does the prize money. The first week of the contest the prize amount is $10, and it doubles each week.
   a. What is the prize amount in the fourth week of the contest? In the tenth week?
   b. What is the total amount of money given away during the entire promotion?

18. In case of a school closing due to inclement weather, the high school staff has a calling system to make certain that everyone is notified. In the first round of phone calls, the principal calls three staff members. In the second round of calls, each of those three staff members calls three more staff members. The process continues until all of the staff has been notified after the fourth round of calls.
   a. Write a rule that shows how many staff members are called during the nth round of calls.
   b. Find the number of staff members called during the fourth round of calls.
   c. If all of the staff has been notified after the fourth round of calls, how many people are on staff at the high school, including the principal?

Lesson 20-3

19. Find the infinite sum if it exists. If it does not exist, tell why.
   a. $24 + 12 + 6 + 3 + \ldots$
   b. $\frac{1}{12} + \frac{1}{6} + \frac{1}{3} + \frac{2}{3} + \ldots$
   c. $1296 - 216 + 36 - 6 + \ldots$

20. Write an expression in terms of $a_n$ that means the same as $\sum_{k=1}^{\infty} 3 \left( \frac{1}{3} \right)^k$

21. Express $0.2727\ldots$ as a fraction.

22. Use the common ratio to determine if the infinite series converges or diverges.
   a. $36 + 24 + 12 + \ldots$
   b. $-4 + 2 + (-1) + \ldots$
   c. $3 + 4.5 + 6.75 + \ldots$

23. The infinite sum $0.1 + 0.05 + 0.025 + 0.0125 + \ldots$
   a. diverges.
   b. converges at 0.2.
   c. converges at 0.5.
   d. converges at 1.0.

24. An infinite geometric series has $a_1 = 3$ and a sum of 4. Find $r$.

25. The graph depicts which of the following?

26. True or false? No arithmetic series with a common difference that is not equal to zero has a limiting value. Explain.

MATHEMATICAL PRACTICES

27. Explain how knowing any two terms of a geometric sequence is sufficient for finding the other terms.
In a classic math problem, a king wants to reward a knight who has rescued him from an attack. The king gives the knight a chessboard and plans to place money on each square. He gives the knight two options. Option 1 is to place a thousand dollars on the first square, two thousand on the second square, three thousand on the third square, and so on. Option 2 is to place one penny on the first square, two pennies on the second, four on the third, and so on.

Think about which offer sounds better and then answer these questions.

1. List the first five terms in the sequences formed by the given options. Identify each sequence as arithmetic, geometric, or neither.
   a. Option 1
   b. Option 2

2. For each option, write a rule that tells how much money is placed on the $n$th square of the chessboard and a rule that tells the total amount of money placed on squares 1 through $n$.
   a. Option 1
   b. Option 2

3. Find the amount of money placed on the 20th square of the chessboard and the total amount placed on squares 1 through 20 for each option.
   a. Option 1
   b. Option 2

4. There are 64 squares on a chessboard. Find the total amount of money placed on the chessboard for each option.
   a. Option 1
   b. Option 2

5. Which gives the better reward, Option 1 or Option 2? Explain why.

Embedded Assessment 1

Assessment Focus
- Identifying terms in arithmetic and geometric sequences
- Identifying common differences and common ratios
- Writing implicit and explicit rules for arithmetic and geometric sequences

Answer Key
1. a. 1000, 2000, 3000, 4000, 5000; arithmetic
   b. 0.01, 0.02, 0.04, 0.08, 0.16; geometric

2. a. $a_n = 1000 + (n - 1)1000$; $S_n = \frac{n}{2}(1000 + a_n)$
   b. $a_n = 0.01(2)^{n-1}$; $S_n = 0.01 \left(1 - \frac{2^n}{2}\right)$ or $0.01(2^n - 1)$

3. a. $\$20,000; \$210,000$
   b. $\$5242.88; \$10,485.75$

4. a. $\$2,080,000$
   b. about $\$1.845 \times 10^{17}$

5. Option 2 is better. The first term in the arithmetic series is much greater than the first term in the geometric series. However, the geometric series grows faster than the arithmetic series. At some point between the 20th and 64th terms, the corresponding terms in the geometric series will be greater than those in the arithmetic series.

Common Core State Standards for Embedded Assessment 1

HSA-SSE.B.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.*

HSF-B.F.A.1 Write a function that describes a relationship between two quantities.*

HSF-B.F.A.1a Determine an explicit expression, a recursive process, or steps for calculation from a context.

HSF-B.F.A.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.*
**Sequences and Series**

**THE CHESSBOARD PROBLEM**

Use after Activity 20

**Embedded Assessment 1**

**Scoring Guide**

<table>
<thead>
<tr>
<th>The solution demonstrates these characteristics:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exemplary</td>
</tr>
</tbody>
</table>

**Mathematics Knowledge and Thinking**

**Items 1, 3, 4**

- **Fluency in determining specified terms of a sequence or the sum of a specific number of terms of a series**
- **A functional understanding and accurate identification of specified terms of a sequence or the sum of a specific number of terms of a series**
- **Partial understanding and partially accurate identification of specified terms of a sequence or the sum of a specific number of terms of a series**
- **Little or no understanding and inaccurate identification of specified terms of a sequence or the sum of a specific number of terms of a series**

**Problem Solving**

**Items 3, 4**

- **An appropriate and efficient strategy that results in a correct answer**
- **A strategy that may include unnecessary steps but results in a correct answer**
- **A strategy that results in some incorrect answers**
- **No clear strategy when solving problems**

**Mathematical Modeling / Representations**

**Items 1, 2**

- **Fluency in accurately representing real-world scenarios with arithmetic and geometric sequences and series**
- **Little difficulty in accurately representing real-world scenarios with arithmetic and geometric sequences and series**
- **Some difficulty in representing real-world scenarios with arithmetic and geometric sequences and series**
- **Significant difficulty in representing real-world scenarios with arithmetic and geometric sequences and series**

**Reasoning and Communication**

**Item 5**

- **Clear and accurate explanation of which option provides the better reward**
- **Adequate explanation of which option provides the better reward**
- **Misleading or confusing explanation of which option provides the better reward**
- **Incomplete or inadequate explanation of which option provides the better reward**

---

**Unpacking Embedded Assessment 2**

Once students have completed this Embedded Assessment, turn to Embedded Assessment 2 and unpack it with them. Use a graphic organizer to help students understand the concepts they will need to know to be successful on Embedded Assessment 2.
Exponential Functions and Graphs

Learning Targets:
- Identify data that grow exponentially.
- Compare the rates of change of linear and exponential data.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Quickwrite

Ramon Hall, a graphic artist, needs to make several different-sized draft versions of an original design. His original graphic design sketch is contained within a rectangle with a width of 4 cm and a length of 6 cm. Using the office copy machine, he magnifies the original 4 cm design to 120% of its new size. Each new draft is 120% of the previous draft.

1. Complete the table with the dimensions of Ramon’s first five draft versions, showing all decimal places.

<table>
<thead>
<tr>
<th>Number of Magnifications</th>
<th>Width (cm)</th>
<th>Length (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>4.8</td>
<td>7.2</td>
</tr>
<tr>
<td>2</td>
<td>5.76</td>
<td>8.64</td>
</tr>
<tr>
<td>3</td>
<td>6.912</td>
<td>10.368</td>
</tr>
<tr>
<td>4</td>
<td>8.2944</td>
<td>12.4416</td>
</tr>
<tr>
<td>5</td>
<td>9.95328</td>
<td>14.92992</td>
</tr>
</tbody>
</table>

2. Make sense of problems. The resulting draft for each magnification has a unique width and a unique length. Thus, there is a functional relationship between the number of magnifications \( n \) and the resulting width \( W \). There is also a functional relationship between the number of magnifications \( n \) and the resulting length \( L \). What are the reasonable domain and range for these functions? Explain.

The reasonable domain for both functions is the nonnegative integers, because \( n \) represents a number of magnifications and thus cannot be negative or a decimal or fraction. The reasonable range for both functions is the nonnegative real numbers, because \( W \) and \( L \) represent measurements and so cannot be negative.

3. Plot the ordered pairs \((n, W)\) from the table in Item 1. Use a different color or symbol to plot the ordered pairs \((n, L)\).

---

ACTIVITY 21

Investigative

Activity Standards Focus

In Activity 21, students examine exponential functions and their graphs. They begin by investigating linear growth and decay and compare rates of change in exponential and linear data. Next, they learn to write exponential functions. They perform transformations of the parent exponential function and, finally, they examine base exponential functions. Students will rely on prior knowledge to investigate rates of change of exponential functions and to perform transformations of the parent function. Review transformations of previous types of functions, including quadratic and cubic.

Lesson 21-1

Pacing: 1 class period

Chunking the Lesson

#1–5    #6–8

Check Your Understanding

Lesson Practice

Common Core State Standards for Activity 21

<table>
<thead>
<tr>
<th>Standard</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>HSF-IF.B.5</td>
<td>Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.</td>
</tr>
<tr>
<td>HSF-IF.B.6</td>
<td>Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.</td>
</tr>
<tr>
<td>HSF-IF.C.7</td>
<td>Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*</td>
</tr>
<tr>
<td>HSF-IF.C.7e</td>
<td>Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.</td>
</tr>
</tbody>
</table>

Differentiating Instruction

Have students who are having difficulty determining whether the points form a line use a ruler to attempt drawing lines through each set of points in Item 3.
ACTIVITY 21

4. Use the data in Item 1 to complete the table.

<table>
<thead>
<tr>
<th>Increase in Number of Magnifications</th>
<th>Change in the Width</th>
<th>Change in the Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 1</td>
<td>4.8 − 4 = 0.8</td>
<td>7.2 − 6 = 1.2</td>
</tr>
<tr>
<td>1 to 2</td>
<td>5.76 − 4.8 = 0.96</td>
<td>8.64 − 7.2 = 1.44</td>
</tr>
<tr>
<td>2 to 3</td>
<td>6.912 − 5.76 = 1.152</td>
<td>10.368 − 8.64 = 1.728</td>
</tr>
<tr>
<td>3 to 4</td>
<td>8.2944 − 6.912 = 1.3824</td>
<td>12.4416 − 10.368 = 2.0736</td>
</tr>
<tr>
<td>4 to 5</td>
<td>9.95328 − 8.2944 = 1.65888</td>
<td>14.92992 − 12.4416 = 2.48832</td>
</tr>
</tbody>
</table>

5. From the graphs in Item 3 and the data in Item 4, do these functions appear to be linear? Explain why or why not.

The functions do not appear to be linear because there is not a constant rate of change and the graphs are not lines.

6. Express regularity in repeated reasoning. Explain why each table below contains data that can be represented by a linear function. Write an equation to show the linear relationship between $x$ and $y$.

a.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-3$</th>
<th>$-1$</th>
<th>$1$</th>
<th>$3$</th>
<th>$5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$8$</td>
<td>$5$</td>
<td>$2$</td>
<td>$-1$</td>
<td>$-4$</td>
</tr>
</tbody>
</table>

Sample answer: The data are linear because there is a constant rate of change; $y$ changes by $-3$ units for every $2$ units of change in $x$. The linear equation is $y = -1.5x + 3.5$.

b.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$2$</th>
<th>$5$</th>
<th>$11$</th>
<th>$17$</th>
<th>$26$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$3$</td>
<td>$7$</td>
<td>$15$</td>
<td>$23$</td>
<td>$35$</td>
</tr>
</tbody>
</table>

Sample answer: The data are linear because the ratios $\frac{\Delta y}{\Delta x}$ are constant, $\frac{\Delta y}{\Delta x} = \frac{4}{3}$, for all pairs of data in the table. The linear equation is $y = \frac{4}{3}x + \frac{1}{3}$.

7. Consider the data in the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$24$</td>
<td>$12$</td>
<td>$6$</td>
<td>$3$</td>
<td>$1.5$</td>
</tr>
</tbody>
</table>

a. Can the data in the table be represented by a linear function? Explain why or why not.

No, the data cannot be represented by a linear function because the $y$-values change by a different amount for each unit change in $x$.

b. Describe any patterns that you see in the consecutive $y$-values. Each $y$-value in the table is $\frac{1}{2}$ of the previous entry.
Lesson 21-1
Exploring Exponential Patterns

8. Consider the data in the table in Item 1. How does the relationship of the data in this table compare to the relationship of the data in the table in Item 7?
Sample answer: Both patterns are formed by multiplying each term by a constant to get the next term. In Item 7 the constant multiplier is \(\frac{3}{2}\), and in Item 1 the constant multiplier is 1.2.

Check Your Understanding

9. Complete the table so that the function represented is a linear function.

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>16</td>
<td>22</td>
<td>_</td>
<td>_</td>
<td>40</td>
</tr>
</tbody>
</table>

10. Reason quantitatively. Explain why the function represented in the table cannot be a linear function.

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>7</td>
<td>12</td>
<td>16</td>
<td>19</td>
<td>21</td>
</tr>
</tbody>
</table>

ASSESS

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 21-1 PRACTICE

Model with mathematics. Determine whether each function is linear or nonlinear. Explain your answers.

11. \(x = \) number of equally sized pans of brownies; \(f(x) = \) number of brownies
12. \(x = \) cost of an item; \(f(x) = \) price you pay in a state with a 6% sales tax
13. \(x = \) number of months; \(f(x) = \) amount of money in a bank account with interest compounded monthly
14. \(x = \) | 2 | 4 | 6 | 8 | 10 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>2.6</td>
<td>3.0</td>
<td>3.8</td>
<td>4.8</td>
<td>6.0</td>
</tr>
</tbody>
</table>
15. \(x = \) | 5 | 10 | 15 | 20 | 25 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>1.25</td>
<td>1.00</td>
<td>0.75</td>
<td>0.50</td>
<td>0.25</td>
</tr>
</tbody>
</table>
16. Identify if there is a constant rate of change or constant multiplier. Determine the rate of change or constant multiplier.

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>6</td>
<td>4.8</td>
<td>3.84</td>
<td>3.072</td>
</tr>
</tbody>
</table>

ACTIVITY 21 Continued

6–8 (continued) Look for a Pattern, Quickwrite, Debriefing

Item 8 returns to the numerical pattern of Item 1. Students should realize that they cannot calculate consecutive \(y\)-values by adding a constant amount each time the \(x\)-value increases by 1.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand the difference between a linear function and an exponential function and can use the information in a table to identify each type of function.

Answers

9.

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>16</td>
<td>22</td>
<td>28</td>
<td>34</td>
<td>40</td>
</tr>
</tbody>
</table>

10. Sample answer: The difference between the first two \(f(x)\) values, 12 and 7, is 5. If the function is linear, then 5 would be the common difference and the following ordered pairs would be (3, 17), (4, 22), and (5, 27), which is not the case.

ADAPT

Check students’ answers to the Lesson Practice to ensure that they can differentiate between linear and nonlinear functions given a description of a domain and its corresponding range. Encourage students to write the functions for Items 11–13. Once they have done so, they can create a table of values or graph the function to help them decide whether it is linear.
ACTIVITY 21 Continued

Lesson 21-2

Learning Targets:
• Identify and write exponential functions.
• Determine the decay factor or growth factor of an exponential function.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Create Representations, Look for a Pattern, Quickwrite, Think-Pair-Share

The data in the tables in Items 7 and 8 of the previous lesson were generated by exponential functions. In the special case when the change in the input variable $x$ is constant, the output variable $y$ of an exponential function changes by a multiplicative constant. For example, in the table in Item 7, the increase in the consecutive $x$-values results from repeatedly adding 1, while the decrease in $y$-values results from repeatedly multiplying by the constant $\frac{1}{2}$, known as the exponential decay factor.

1. In the table in Item 1 in Lesson 21-1, what is the exponential growth factor?
   $1.2$

2. You can write an equation for the exponential function relating $W$ and $n$.
   a. Complete the table below to show the calculations used to find the width of each magnification.
   
<table>
<thead>
<tr>
<th>Number of Magnifications</th>
<th>Calculation to Find Width (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$4(1.2)$</td>
</tr>
<tr>
<td>2</td>
<td>$4(1.2)(1.2)$</td>
</tr>
<tr>
<td>3</td>
<td>$4(1.2)(1.2)(1.2)$</td>
</tr>
<tr>
<td>4</td>
<td>$4(1.2)(1.2)(1.2)(1.2)$</td>
</tr>
<tr>
<td>5</td>
<td>$4(1.2)(1.2)(1.2)(1.2)(1.2)$</td>
</tr>
<tr>
<td>$n$</td>
<td>$4(1.2)^n$</td>
</tr>
</tbody>
</table>

   b. Express regularity in repeated reasoning. Write a function that expresses the resulting width $W$ after $n$ magnifications of 120%.
   
   $W(n) = 4(1.2)^n$

   c. Use the function in part b to find the width of the 11th magnification.

   $W(11) \approx 28.720$

Developing Math Language

Review exponential expressions and compare and contrast them to exponential functions. Discuss the difference between growth and decay to help students understand the meanings of exponential growth factors and exponential decay factors.

Teacher to Teacher

When students use the Connect to Technology, the issue of domain may come up. If it does not come up here, revisit it in Item 2 of Lesson 21-3. The domain of the problem situation is a subset of the nonnegative integers. The functions graphed on the calculator will have a domain of all real numbers.

Technology Tip

Students without a graphing calculator may evaluate the expressions in the Calculation to Find Width column for 0 to 5 magnifications to confirm that the function is reasonable. For additional technology resources, visit SpringBoard Digital.
Lesson 21–2
Exponential Functions

The general form of an exponential function is $f(x) = a(b^x)$, where $a$ and $b$ are constants and $a \neq 0$, $b > 0$, $b \neq 1$.

3. For the exponential function written in Item 2b, identify the value of the parameters $a$ and $b$. Then explain their meaning in terms of the problem situation.

   $a$ represents the initial width of 4 cm, and $b$ represents the growth factor of 1.2.

4. Starting with Ramon's original $4 \times 6$ cm rectangle containing his graphic design, write an exponential function that expresses the resulting length $L$ after $n$ magnifications of 120%.

   $L(n) = 6(1.2)^n$

Ramon decides to print five different reduced draft copies of his original design rectangle. Each one will be reduced to 90% of the previous size.

5. Complete the table below to show the dimensions of the first five draft versions. Include all decimal places.

<table>
<thead>
<tr>
<th>Number of Reductions</th>
<th>Width (cm)</th>
<th>Length (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>3.6</td>
<td>5.4</td>
</tr>
<tr>
<td>2</td>
<td>3.24</td>
<td>4.86</td>
</tr>
<tr>
<td>3</td>
<td>2.916</td>
<td>4.374</td>
</tr>
<tr>
<td>4</td>
<td>2.6244</td>
<td>3.9366</td>
</tr>
<tr>
<td>5</td>
<td>2.38196</td>
<td>3.54294</td>
</tr>
</tbody>
</table>

6. Write the exponential decay factor and the decay rate for the data in the table in Item 5.

   The exponential decay factor is 0.9 and the decay rate is 10%.

7. Model with mathematics. Use the data in the table in Item 5.

   a. Write an exponential function that expresses the width $w$ of a reduction in terms of $n$, the number of reductions performed.

   $w(n) = 4(0.9)^n$

   b. Write an exponential function that expresses the length $l$ of a reduction in terms of $n$, the number of reductions performed.

   $l(n) = 6(0.9)^n$

   c. Use the functions to find the dimensions of the design if the original design undergoes ten reductions.

   $w(n) \approx 1.395$; $l(n) \approx 2.092$

To compare change in size, you could also use the decay rate, or percent decrease. This is the percent that is equal to the ratio of the decrease amount to the original amount.

ACTIVITY 21

3–4 Quickwrite, Create Representations, Debriefing

In Item 3, students will explain the meaning of parameters of the equation in terms of the problem situation.

Some students may extend their response in Item 3 to a new resizing situation and simply write the function using $a = 6$ cm and $b = 1.2$. Other students may replicate the work in Item 2 by making a table of values.

Differentiating Instruction

To support students’ writing efforts, you may want to add a section to your Word Wall on basic sentence structure in English writing (simple sentence, compound sentence, complex sentence, transition words, etc.). Review these structures with students prior to the writing assignment, and provide an opportunity to clarify any questions about language structures.

5–6 Create Representations, Vocabulary Organizer, Think-Pair-Share

In Item 5, check to see that students follow instructions and do not round numbers in their computations. The intent of Item 6 is to distinguish between decay factor and decay rate. The Math Tip can help guide the students.

5–6 Create Representations, Think-Pair-Share, Debriefing

Students may extend their response in Item 3 to a new resizing situation and simply write the functions using the appropriate values for $a$ and $b$.

Teacher to Teacher

In this section of the activity, students investigate exponential decay in the context of the original problem. Instead of repeated magnifications of 120%, this second considers repeated reductions of 90%.

7 Create Representations, Think-Pair-Share, Debriefing

Students may extend their response in Item 3 to a new resizing situation and simply write the functions using the appropriate values for $a$ and $b$. 

ACTIVITY 21 Continued
**Check Your Understanding**

Debrief students’ answers to these items to ensure that they can write an exponential function and identify the meaning of \( a \) and \( b \) in context.

**Answers**

8. Sample answer: If \( a = 0 \), the function would be \( f(x) = 0 \), a constant function. If \( b = 1 \), the function would be \( f(x) = 1 \), also a constant function. If \( b < 0 \), the function would not be continuous.

9. a. 2  
b. Exponential growth; The function values are increasing.

10. a. \( a = 2000; \) \( b = 1.05 \); $2000 is deposited in an account with an annual interest rate of 5%.

**LESSON 21-2 PRACTICE**

Construct viable arguments. Decide whether each table of data can be modeled by a linear function, an exponential function, or neither, and justify your answers. If the data can be modeled by a linear or exponential function, give an equation for the function.

11. Exponential; \( x \) increases by a constant amount while \( y \) increases at a rate of \( 3 \); \( y = 3^x \).

12. Neither; \( x \) increases by a constant amount; \( y \) increases, but not by a constant amount or a constant multiplier.

13. Given that the function has an exponential decay factor of 0.8, complete the table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>64</td>
<td>51.2</td>
<td>40.96</td>
<td>32.768</td>
<td>26.2144</td>
</tr>
</tbody>
</table>

14. What is the decay rate for the function in Item 13?

15. Write the function represented in Item 13.
Graph the functions and determine the domain and range for each function. Use interval notation.

A function is said to increase when the y-values increase as the x-values increase. A function is said to decrease if the y-values decrease as the x-values increase.

Describe the end behavior of each function as x approaches ±∞.

Not all functions increase or decrease over the entire domain of the function. Functions may increase, decrease, or remain constant over various intervals of the domain. Functions that either increase or decrease over the entire domain are called strictly monotonic.

SUGGESTED LEARNING STRATEGIES: Create Representations, Activating Prior Knowledge, Close Reading, Vocabulary Organizer, Think-Pair-Share, Group Presentation

1. Graph the functions \( y = 6(1.2)^x \) and \( y = 6(0.9)^x \) on a graphing calculator or another graphing utility. Sketch the results.

2. Determine the domain and range for each function. Use interval notation.

   - a. \( y = 6(1.2)^x \)
     - Domain: \( (-\infty, \infty) \)
     - Range: \( (0, \infty) \)

   - b. \( y = 6(0.9)^x \)
     - Domain: \( (-\infty, \infty) \)
     - Range: \( (0, \infty) \)

3. Describe each function as increasing or decreasing.

   - a. \( y = 6(1.2)^x \)
     - Increasing

   - b. \( y = 6(0.9)^x \)
     - Decreasing

As you learned in a previous activity, the end behavior of a graph describes the y-values of the function as x increases without bound and as x decreases without bound. If the end behavior approaches some constant a, then the graph of the function has a horizontal asymptote at \( y = a \).

When x increases without bound, the values of x approach positive infinity, \( \infty \). When x decreases without bound, the values of x approach negative infinity, \( -\infty \).

4. Describe the end behavior of each function as x approaches \( \infty \). Write the equation for any horizontal asymptotes.

   - a. \( y = 6(1.2)^x \)
     - As x goes to infinity, y goes to infinity.

   - b. \( y = 6(0.9)^x \)
     - As x goes to infinity, y gets close to 0; there is a horizontal asymptote at \( y = 0 \).

Activating Prior Knowledge, Debriefing

Students should have an intuitive knowledge of when a function is increasing and when it is decreasing. Ask students to describe what the graph of an increasing function looks like and what the graph of a decreasing function looks like.

Differentiating Instruction

For advanced learners who wish to further explore the Connect to AP, ask students to identify the intervals on which each function shown below is increasing, decreasing, and/or constant. Students should also identify any functions that are strictly monotonic.

- Increasing for all reals; strictly monotonic
- Decreasing for \( x < 0 \); constant for \( 0 < x < 2 \); increasing for \( x > 2 \)
- Decreasing for \( -2 < x < 2 \); increasing for \( x < -2 \) and \( x > 2 \)
ACTIVITY 21 Continued

4–7 Create Representations, Think-Pair-Share, Group Presentation, Debriefing Use Items 4–7 to assess student understanding of the concepts of end behavior, $\infty$, $-\infty$, and asymptotes as related to the graphs of exponential functions. Students should be able to answer Item 6 by referring to the graphs of the functions.

Item 7 is intended to spark a class discussion of the features of the graphs of exponential functions as related to the parameters of the function. Some students may not think about the possibility of the value of $a$ being negative. Try to find a group to consider this and have them present their findings to the class.

Check Your Understanding
Debrief students’ answers to these items to ensure that they understand the meaning of increasing and decreasing and can identify the domain, range, and asymptotes of an exponential function.

Answers
8. domain (both): $(-\infty, \infty)$; range (both): $(-\infty, 0)$
9. $f(x)$: $y$ approaches $-\infty$; $g(x)$: $y$ approaches 0
10. $f(x)$: $y$ approaches 0; $g(x)$: $y$ approaches $-\infty$

ASSESS
Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 21-3 PRACTICE

Make use of structure. For each exponential function, state whether the function increases or decreases, and give the $y$-intercept. Use the general form of an exponential function to explain your answers.

11. $y = (4)^x$  
12. $y = 0.3(0.25)^x$ 
13. $y = -2(1.6)^x$  
14. $y = -(0.3)^x$ 
15. Construct viable arguments. What is true about the asymptotes and $y$-intercepts of the functions in this lesson? What conclusions can you draw?

16. Describe the end behavior of each function as $x$ approaches $-\infty$. Write the equation for any horizontal asymptotes.
   a. $y = 6(1.2)^x$  
      As $x$ goes to negative infinity, $y$ gets close to 0; there is a horizontal asymptote at $y = 0$.
   b. $y = 6(0.9)^x$  
      As $x$ goes to negative infinity, $y$ goes to infinity.

17. Identify any $x$- or $y$-intercepts of each function.
   a. $y = 6(1.2)^x$  
      $x$-intercept: none; $y$-intercept: $(0, 6)$
   b. $y = 6(0.9)^x$  
      $x$-intercept: none; $y$-intercept: $(0, 6)$

18. Reason abstractly. Consider how the parameters $a$ and $b$ affect the graph of the general exponential function $f(x) = ab^x$. Use a graphing calculator to graph $f$ for various values of $a$ and $b$.
   a. When does the function increase?  
      for $a > 0$ and $b > 1$ or for $a < 0$ and $0 < b < 1$
   b. When does the function decrease?  
      for $a > 0$ and $0 < b < 1$ or for $a < 0$ and $b > 1$
   c. What determines the $y$-intercept of the function?  
      the value of $a$
   d. State any horizontal asymptotes of the function.  
      $y = 0$

Check Your Understanding
Graph the functions $f(x) = -6(1.2)^x$ and $g(x) = -6(0.9)^x$ on a graphing calculator or other graphing utility.

8. Determine the domain and range for each function.
9. Describe the end behavior of each function as $x$ approaches $\infty$.
10. Describe the end behavior of each function as $x$ approaches $-\infty$.

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You can use transformations of the graph of the function $f(x) = b^x$ to graph functions of the form $g(x) = ab^{x-h} + d$, where $a$ and $b$ are constants, and $a < 0$, $b > 0$, $b = 1$. Rather than having a single parent graph for all exponential functions, there is a different parent graph for each base $b$.

1. Graph the parent graph $f$ and the function $g$ by applying the correct transformation of the graph of the parent graph for any base $b$ by plotting the points $(-1, \frac{1}{b})$, $(0, 1)$, and $(1, b)$.

### Learning Targets:
- Explore how changing parameters affects the graph of an exponential function.
- Graph transformations of exponential functions.

### SUGGESTED LEARNING STRATEGIES:
- Close Reading
- Create Representations
- Quickwrite

### MATH TIP
Exponential functions are important in the study of calculus.

You can draw a quick sketch of the graph of the parent graph for any base $b$ by plotting the points $(-1, \frac{1}{b})$, $(0, 1)$, and $(1, b)$.

### Activity 21
**Bell-Ringer Activity**
Ask students to describe each transformation of the graph of the parent function $f(x) = x^2$.

1. $g(x) = (x + 3)^2 - 1$ [shifted 3 units left and 1 unit down]
2. $g(x) = -2(x - 2)^2$ [reflection across $x$-axis and a vertical stretch by a factor of 2]
3. $g(x) = 3(x - 1)^2$ [vertical stretch by a factor of 3 and shifted 1 unit right]

### Debriefing
Students should see that the constant $a$ can cause a vertical stretch or shrink of the parent graph. When $a < 0$, there is a reflection over the $x$-axis.

### Connect to AP
Exponential functions play an important part in the study of calculus and have some very interesting properties. The instantaneous rate of change (derivative) of any exponential function in the form $f(x) = b^x$ is a multiple of the function $f(x)$. This makes sense when one considers that the slope of an exponential function is always increasing at an increasing rate as $x$ increases. A function that fits this description well is an exponential function. Even more surprising is that the derivative of the exponential function $f(x) = e^x$—a special exponential function that is discussed in the next lesson—is the function itself. This is the only nonzero function that has the property of being its own derivative.
2. Sketch the parent graph \( f \) and the graph of \( g \) by applying the correct horizontal or vertical translation. Write a description of each transformation and give the equations of any asymptotes.

a. \( f(x) = 2^x \)
\( g(x) = 2^{x-3} \)

b. \( f(x) = \left(\frac{1}{3}\right)^x \)
\( g(x) = \left(\frac{1}{3}\right)^x - 2 \)

**Technology Tip**

Discuss asymptotic behavior as it relates to graphs of exponential functions. Explain that in Item 2a, it looks like the graph touches the \( x \)-axis but, in fact, it never does. Have students graph the function using an online calculator or geometry software. Have them zoom in to the \( x \)-axis, where they will see that the function does not cross or intersect it.
3. **Attend to precision.** Describe how each function results from transforming a parent graph of the form $f(x) = b^x$. Then sketch the parent graph and the given function on the same axes. Give the domain and range of each function in interval notation. Give the equations of any asymptotes.

a. $g(x) = 3^{x+4} + 1$

To obtain the graph of $g$, horizontally translate the graph of $f$ left 4 units and then vertically translate up 1 unit.

- **Domain:** $(-\infty, \infty)$
- **Range:** $(0, \infty)$
- **Asymptotes:** $y = 0, y = 1$

b. $g(x) = 2^{(\frac{1}{3})^x} - 4$

To obtain the graph of $g$, vertically stretch the graph of $f$ by a factor of 2 and then vertically translate down 4 units.

- **Domain:** $(-\infty, \infty)$
- **Range:** $(0, \infty)$
- **Asymptotes:** $y = 0, y = -4$
7. The parent graph is \( f(x) = 4^x \); there is a horizontal translation 2 units to the right and a shift down 3 units; domain: \((-\infty, \infty)\); range: \((-3, \infty)\); asymptote: \( y = -3 \).

8. The parent graph is \( f(x) = 2^x \); there is a vertical shrink of \( \frac{1}{2} \), a horizontal translation 4 units to the right, and a vertical shift up 1 unit; domain: \((-\infty, \infty)\); range: \((1, \infty)\); asymptote: \( y = 1 \).

9. The parent graph is \( f(x) = \left(\frac{1}{3}\right)^x \); there is a reflection over the \( x \)-axis, a vertical stretch by a factor of 2, and a horizontal translation left 4 units; vertical shift up 1 unit; domain: \((-\infty, \infty)\); range: \((-\infty, 0)\); asymptote: \( y = 0 \).

c. \( g(x) = \frac{1}{2}(4)^{x-4} - 2 \)

To obtain the graph of \( g \), vertically shrink the graph of \( f \) by a factor of \( \frac{1}{2} \), translate horizontally to the right 4 units, and then translate vertically down 2 units.

\[ f \quad \text{Domain:} \quad (-\infty, \infty) \quad \text{Range:} \quad (0, \infty) \quad \text{Asymptotes:} \quad y = 0 \]

\[ g \quad \text{Domain:} \quad (-\infty, \infty) \quad \text{Range:} \quad (-\infty, \infty) \quad \text{Asymptotes:} \quad y = -2 \]

4. Describe how the function \( g(x) = -3(2)^{x-6} + 5 \) results from transforming a parent graph \( f(x) = 2^x \). Sketch both graphs on the same axes. Give the domain and range of each function in interval notation. Give the equations of any asymptotes. Use a graphing calculator to check your work.

To obtain the graph of \( g \), reflect the graph of \( f \) over the \( x \)-axis, vertically stretch the graph of \( f \) by a factor of 3, and then vertically translate up 5 units and horizontally translate 6 units to the right.
Check Your Understanding

5. **Reason quantitatively.** Explain how to change the equation of a parent graph \( f(x) = 4^x \) to a translation that is left 6 units and a vertical shrink of 0.5.

6. Write the parent function \( f(x) \) of \( g(x) = -3(2)^{x+2} - 1 \) and describe how the graph of \( g(x) \) is a translation of the parent function.

**LESSON 21-4 PRACTICE**

Describe how each function results from transforming a parent graph of the form \( f(x) = b^x \). Then sketch the parent graph and the given function on the same axes. State the domain and range of each function and give the equations of any asymptotes.

7. \( g(x) = 4^{x-2} - 3 \)

8. \( g(x) = \frac{1}{2}(2)^{x+4} + 1 \)

9. \( g(x) = -2\left(\frac{1}{3}\right)^{x+4} \)

**Make use of structure.** Write the equation that indicates each transformation of the parent equation \( f(x) = 2^x \). Then use the graph below and draw and label each transformation.

10. For \( g(x) \), the \( y \)-intercept is at \((0, 3)\).

11. For \( h(x) \), the exponential growth factor is 0.5.

12. For \( k(x) \), the graph of \( f(x) \) is horizontally translated to the right 3 units.

13. For \( l(x) \), the graph of \( f(x) \) is vertically translated upward 2 units.

**ASSESS**

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

**LESSON 21-4 PRACTICE**

7–9. See page 334.

10. \( g(x) = 3(2)^x \)

11. \( h(x) = (0.5)^x \)

12. \( k(x) = (2)^{x-3} \)

13. \( l(x) = (2)^x + 2 \)

**ADAPT**

Check students’ answers to the Lesson Practice to ensure that they can identify attributes of a transformation from an equation. Encourage students to graph the functions in Items 7–9 to check their answers.
Learning Targets:
• Graph the function \( f(x) = e^x \).
• Graph transformations of \( f(x) = e^x \).

SUGGESTED LEARNING STRATEGIES: Quickwrite, Group Presentation, Debriefing

1. Use appropriate tools strategically. On a graphing calculator, set \( Y_1 = x \) and \( Y_2 = (1 + \frac{1}{x})^x \). Let \( x \) increase by increments of 100. Describe what happens to the table of values for \( Y_2 \) as \( x \) increases.

Sample answer: The values of \( Y_2 \) approach a constant value, \( \approx 2.7183 \).

This irrational constant is called \( e \) and is often used in exponential functions.

2. a. On a graphing calculator, enter \( Y_1 = e^x \). Using the table of values associated with \( Y_1 \), complete the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( Y_1 = e^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2.7183</td>
</tr>
<tr>
<td>2</td>
<td>7.3891</td>
</tr>
<tr>
<td>3</td>
<td>20.086</td>
</tr>
</tbody>
</table>

b. Reason quantitatively. Which row in the table gives the approximate value of \( e \)? Explain.

The row in which \( x = 1 \); when \( x = 1, Y_1 = e^1 = e \).

c. What kind of number does \( e \) represent?

an irrational number

3. a. Complete the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x^{-1} )</th>
<th>( x^0 )</th>
<th>( x^1 )</th>
<th>( x^2 )</th>
<th>( x^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.5</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>( e )</td>
<td>0.3679</td>
<td>1</td>
<td>2.7183</td>
<td>7.3891</td>
<td>20.086</td>
</tr>
<tr>
<td>3</td>
<td>0.3333</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>27</td>
</tr>
</tbody>
</table>

b. Graph the functions \( f(x) = e^x \), \( g(x) = 2^x \), and \( h(x) = 3^x \) on the same coordinate plane.

c. Compare \( f(x) \) with \( g(x) \) and \( h(x) \). Which features are the same? Which are different?

All three functions have the \( y \)-intercept \((0, 1)\) and the horizontal asymptote \( y = 0 \). The graph of the function \( f(x) \) is between the graphs of \( g(x) \) and \( h(x) \), which makes sense because \( 2 < e < 3 \).
4. Graph the parent function \( f(x) = e^x \) and the function \( g(x) \) by applying the correct vertical stretch, shrink, reflection over the x-axis, or translation. Write a description for the transformation. State the domain and range of each function. Give the equation of any asymptotes.

a. \( f(x) = e^x \)
   \[ g(x) = -\frac{1}{2}e^x \]
   The graph of \( g \) is a vertical shrink of the graph of \( f \) by a factor of \( \frac{1}{2} \) and a reflection over the x-axis. The domain of both functions is \((−\infty, \infty)\). The range of \( f \) is \((0, \infty)\); the range of \( g \) is \((−\infty, 0)\). The asymptote of both functions is \( y = 0 \).

b. \( f(x) = e^x \)
   \[ g(x) = e^{x-4} + 1 \]
   The graph of \( g \) is a horizontal translation to the right 4 units and a vertical translation up 1 unit of \( f \). The domain of both functions is \((−\infty, \infty)\). The range of \( f \) is \((0, \infty)\); the range of \( g \) is \((1, \infty)\). The asymptote of \( f \) is \( y = 0 \); the asymptote of \( g \) is \( y = 1 \).
5. Graph the parent graph $f$ and the function $g$ by applying the correct transformation. Write a description of each transformation. State the domain and range of each function. Give the equation of any asymptotes.

a. $f(x) = e^x$
   
   $g(x) = 2e^x - 5$

   The graph of $g$ is a vertical stretch of the graph of $f$ by a factor of 2 and a vertical translation down 5 units. The domain of both functions is $(-\infty, \infty)$. The range of $f$ is $(0, \infty)$; the range of $g$ is $(-5, \infty)$. The asymptote of $f$ is $y = 0$; the asymptote of $g$ is $y = -5$.

b. $f(x) = e^x$
   
   $g(x) = \frac{1}{2}(e^{x-4}) - 2$

   The graph of $g$ is a vertical shrink of the graph of $f$ by a factor of $\frac{1}{2}$, a horizontal translation right 4 units, and a vertical translation down 2 units. The domain of both functions is $(-\infty, \infty)$. The range of $f$ is $(0, \infty)$; the range of $g$ is $(-2, \infty)$. The asymptote of $f$ is $y = 0$; the asymptote of $g$ is $y = -2$.
6. Explain how the parameters $a$, $c$, and $d$ transform the parent graph $f(x) = b^x$ to produce the graph of the function $g(x) = a(b)^{x-c} + d$.

- $|a| > 1 \Rightarrow$ a vertical stretch by a factor of $a$
- $0 < |a| < 1 \Rightarrow$ a vertical shrink by a factor of $a$
- $a < 0 \Rightarrow$ a reflection over the $x$-axis
- $c > 0 \Rightarrow$ a horizontal translation right $c$ units
- $c < 0 \Rightarrow$ a horizontal translation left $c$ units
- $d < 0 \Rightarrow$ a vertical shift down $d$ units
- $d > 0 \Rightarrow$ a vertical shift up $d$ units

### Check Your Understanding

Match each exponential expression with its graph.

7. $f(x) = 3e^x$

8. $f(x) = -0.4e^x$

9. $f(x) = e^x + 2$

10. $f(x) = -e^x$

**Answers**

7. D  
8. B  
9. A  
10. C
LESSON 21-5 PRACTICE

Model with mathematics. Describe how each function results from transforming a parent graph of the form \( f(x) = e^x \). Then sketch the parent graph and the given function on the same axes. State the domain and range of each function and give the equations of any asymptotes.

11. \( g(x) = \frac{1}{4}e^x + 5 \)
12. \( g(x) = e^{x-3} - 4 \)
13. \( g(x) = -4e^{x-3} + 3 \)
14. \( g(x) = 2e^{x+4} \)

15. Critique the reasoning of others. On Cameron’s math test, he was asked to describe the transformations from the graph of \( f(x) = e^x \) to the graph of \( g(x) = e^{x-2} - 2 \). Cameron wrote “translation left 2 units and down 2 units.” Do you agree or disagree with Cameron? Explain your reasoning.

16. What similarities, if any, are there between the functions studied in this lesson and the previous lesson?
ACTIVITY 21 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 21-1

1. a. Complete the table so that the function represented is a linear function.

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>5.4</td>
<td>6.7</td>
<td>10.6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. What function is represented in the data?

2. a. How do you use a table of values to determine if the relationship of $y = 3x + 2$ is a linear relationship?
b. How do you use a graph to determine if the relationship in part a is linear?

3. Which relationship is nonlinear?
   A. $(2, 12), (5, 18), (6, 5, 21)$
   B. $(6, x + 2), (21, x + 7), (−9, x − 3)
   C. $(0.25, 1.25), (1.25, 2.50), (2.50, 5.00)$
   D. $(-5, 20), (-3, 12), (-1, 4)$

4. Determine if the table of data can be modeled by a linear function. If so, give an equation for the function. If not, explain why not.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{2}{5}$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{5}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{4}{5}$</td>
</tr>
</tbody>
</table>

5. Which relationship has the greatest value for $x = 4$?
   A. $y = 5(3^x) + 2$
   B. $y = 5(2^x) + 3$
   C. $y = 5(3x) + 2$
   D. $y = 5(2^{x-3}$

6. Ida paints violets onto porcelain plates. She paints a spiral that is a sequence of violets, the size of each consecutive violet being a fraction of the size of the preceding violet. The table below shows the width of the first three violets in the continuing pattern.

<table>
<thead>
<tr>
<th>Violet Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width (cm)</td>
<td>4</td>
<td>3.2</td>
<td>2.56</td>
</tr>
</tbody>
</table>

a. Is Ida’s shrinking violet pattern an example of an exponential function? Explain.
b. Find the width of the fourth and fifth violets in the sequence.

Lesson 21-2

7. Which statement is NOT true for the exponential function $f(x) = 4(0.75)^x$?
   A. Exponential growth factor is 75%.
   B. Percent of decrease is 25%.
   C. The scale factor is 0.75.
   D. The decay rate is 25%.

8. For the exponential function $f(x) = 3(1.5)^x$, identify the value of the parameters $a$ and $b$. Then explain their meaning, using the vocabulary from the lesson.

9. Decide whether each table of data can be modeled by a linear function, an exponential function, or neither. If the data can be modeled by a linear or exponential function, give an equation for the function.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>24</td>
<td>18</td>
<td>12</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

b. | x  | 0    | 1    | 2    | 3    | 4    |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>36</td>
<td>18</td>
<td>9</td>
<td>4.5</td>
<td>2.25</td>
</tr>
</tbody>
</table>

10. Sixteen teams play in a one-game elimination match. The winners of the first round go on to play a second round until only one team remains undefeated and is declared the champion.

a. Make a table of values for the number of rounds and the number of teams participating. 
   b. What is the reasonable domain and the range of this function? Explain.
   c. Find the rate of decay.
   d. Find the decay factor.

LESSON 21-5 PRACTICE (continued)

15. The correct answer is “translation right 2 units and down 2 units.” Cameron likely made the error of thinking that subtracting 2 in the exponent results in a translation to the left because, on a number line, subtraction is usually associated with moving to the left.

16. Since $f(x) = e^x$ is an exponential function, the functions in both lessons are exponential functions and follow the same rules.

ACTIVITY 21 Continued

ACTIVITY 21 PRACTICE

1. a. $f(x) = 5.4$
2. a. Sample answer: Subtract consecutive $y$-values to determine if there is a common difference.
b. Sample answer: The graph is a straight line.
3. C
4. yes; $y = \frac{2}{5}x + \frac{1}{5}$
5. D
6. a. Yes, it is an example of an exponential function because as the $x$-values increase by 1 each time, there is a constant ratio of 0.8 between the $y$-values.
   b. 2.56(0.8) = 2.048 cm;
   c. $y = (0.8)^{x-1}$ or $y = 5(0.8)^x$
d. You can substitute 10 into the function given in part c and have $y = 4(0.8)^{10-1} = 0.537$ cm rounded to three decimal places, or you can use a calculator and repeatedly multiply the starting violet width by 0.8 and find the same result.

7. A
8. $a = 3.2$, the initial value; $b = 1.5$, the growth factor or scale factor
9. a. linear; $y = -6x + 24$
b. exponential; $y = 36\left(\frac{1}{2}\right)^x$

10. a. $f(x) = 16$
    b. domain: $(x): 1, 2, 3, 4, 5$; range: $(y): 16, 8, 4, 2, 1$
    c. 50%
    d. 0.50

11. A and C
12. B
13. a. $(\infty, \infty); (0, \infty)$; increases; $(0, 2)$
    b. $(\infty, \infty); (0, \infty)$; decreases; $(0, 3)$
    c. $(\infty, \infty); (\infty, 0)$; increases; $(0, -1)$
    d. $(\infty, \infty); (\infty, 0)$; decreases; $(0, -3)$
14. a. $1.37\%: 0.84\%; 0.362\%$
    b. $1.37\%: 0.84\% + 0.362\% = 0.892\%$
    c. 100.892%
    d. $P(n) = 313,847,465(1.00892)^n$, where $P$ is population and $n = \text{years since 2012}$
    e. $P(38) \approx 439,819,438$
15. $f(x)$ is increasing when $a > 0$. 

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ACTIVITY 21 Continued

16. a. 

Translate the graph of \( f \) horizontally to the left 3 units and then vertically translate down 4 units. 
\[ f: \text{domain } (-\infty, \infty); \text{range } (0, \infty); \text{asymptote: } y = 0 \]
\[ g: \text{domain } (-\infty, \infty); \text{range } (-4, \infty); \text{asymptote: } y = -4 \]

b. 

Vertically stretch the graph of \( f \) by a factor of 3, reflect over the x-axis, and vertically translate up 2 units. 
\[ f: \text{domain } (-\infty, \infty); \text{range } (0, \infty); \text{asymptote: } y = 0 \]
\[ g: \text{domain } (-\infty, \infty); \text{range } (-\infty, 2); \text{asymptote: } y = 2 \]

c. 

Vertically shrink the graph of \( f \) by a factor of \( \frac{1}{2} \), horizontally translate left 3 units, and then vertically translate down 4 units. 
\[ f: \text{domain } (-\infty, \infty); \text{range } (0, \infty); \text{asymptote: } y = 0 \]
\[ g: \text{domain } (-\infty, \infty); \text{range } (-4, \infty); \text{asymptote: } y = -4 \]

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

ACTIVITY 21 continued

Lesson 21-3

11. Which of the following functions have the same graph?
   A. \( f(x) = \left(\frac{1}{2}\right)^x \)
   B. \( f(x) = 4^x \)
   C. \( f(x) = 4^{-x} \)
   D. \( f(x) = x^4 \)

12. Which function is modeled in the graph below?

   \[ A. y = (2)^x \quad B. y = 2(1.1)^x \]
   \[ C. y = (2)^{1.1} \quad D. y = 2.1x \]

13. For each exponential function, state the domain and range, whether the function increases or decreases, and the y-intercept.
   a. \( y = 2(4)^x \)
   b. \( y = 3\left(\frac{1}{2}\right)^x \)
   c. \( y = -0.3^x \)
   d. \( y = -3(5.2)^x \)

14. The World Factbook produced by the Central Intelligence Agency estimates the July 2012 United States population as 313,847,465. The following rates are also reported as estimates for 2012.
   Birth rate: 13.7 births/1000 population
   Death rate: 8.4 deaths/1000 population
   Net migration rate: 3.62 migrant(s)/1000 population
   a. Write a percent for each rate listed above.
   b. Combine the percents from part a to find the overall growth rate for the United States.
   c. The exponential growth factor for a population is equal to the growth rate plus 100%. What is the exponential growth rate for the United States?
   d. Write a function to express the United States population as a function of years since 2012.
   e. Use the function from part d to predict the United States population in the year 2050.

15. Under what conditions is the function \( f(x) = a(3)^x \) increasing?

Lesson 21-4

16. Describe how each function results from transforming a parent graph of the form \( f(x) = b^x \). Then sketch the parent graph and the given function on the same axes. State the domain and range of each function and give the equations of any asymptotes.
   a. \( g(x) = 2^{x+3} - 4 \)
   b. \( g(x) = -3\left(\frac{1}{2}\right)^x + 2 \)
   c. \( g(x) = \frac{1}{2}(3)^{x+3} - 4 \)

17. a. Explain why a change in \( c \) for the function \( a(b)^{-x} + d \) causes a vertical translation.
   b. Explain why a change in \( d \) for the function \( a(b)^{x+c} + d \) causes a vertical translation.

18. Which transformation maps the graph of \( f(x) = 3^x \) to \( g(x) = \left(\frac{1}{3}\right)^x \)?
   a. horizontal translation
   b. shrink
   c. reflection
   d. vertical translation

Lesson 21-5

19. Is \( f(x) = e^x \) an increasing or a decreasing function? Explain your reasoning.

20. Which function has a y-intercept of (0, 0)?
   A. \( y = e^x + 1 \)
   B. \( y = -e^x + 1 \)
   C. \( y = e^x - 1 \)
   D. \( y = e^x \)

21. What ordered pair do \( f(x) = e^x \) and \( g(x) = 2^x \) have in common?

MATHMATICAL PRACTICES

Attend to Precision

22. Explain the difference between \( y = x^2 \) and \( y = 2^x \).

17. a. Sample answer: By adding to or subtracting from the x-value of the exponential term, the exponential term is being evaluated for a different value than \( x \).
   b. Sample answer: By adding to or subtracting from the exponential term containing \( x \), the value of the function will be increased or decreased.

18. C

19. increasing; because \( e > 1 \)

20. B

21. (0, 1)

22. Sample answer: The first is a polynomial function with a constant exponent and a changing base whose graph is a parabola. The second is an exponential function with a constant base and a changing exponent whose graph is nonlinear but is not a parabola.

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In 1935, Charles F. Richter developed the Richter magnitude test scale to compare the size of earthquakes. The Richter scale is based on the amplitude of the seismic waves recorded on seismographs at various locations after being adjusted for distance from the epicenter of the earthquake. Richter assigned a magnitude of 0 to an earthquake whose amplitude on a seismograph is 1 micron, or $10^{-4}$ cm. According to the Richter scale, an earthquake of magnitude 1.0 causes 10 times the ground motion of a magnitude 0 earthquake. A magnitude 2.0 earthquake causes 10 times the ground motion of a magnitude 1.0 earthquake. This pattern continues as the magnitude of the earthquake increases.

1. **Reason quantitatively.** How does the ground motion caused by earthquakes of these magnitudes compare?
   - a. magnitude 5.0 earthquake compared to magnitude 4.0
     A magnitude 5.0 earthquake's ground motion is 10 times that of a magnitude 4.0 earthquake.
   - b. magnitude 4.0 earthquake compared to magnitude 1.0
     A magnitude 4.0 earthquake's ground motion is 1000 or $10^3$ times that of a magnitude 1.0 earthquake.
   - c. magnitude 4.0 earthquake compared to magnitude 0
     A magnitude 4.0 earthquake's ground motion is $10,000$ or $10^4$ times that of a magnitude 0 earthquake.

The table below describes the effects of earthquakes of different magnitudes.

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>Very weak, no visible damage</td>
</tr>
<tr>
<td>2.0</td>
<td>Not felt by humans</td>
</tr>
<tr>
<td>3.0</td>
<td>Often felt, usually no damage</td>
</tr>
<tr>
<td>4.0</td>
<td>Windows rattle, indoor items shake</td>
</tr>
<tr>
<td>5.0</td>
<td>Damage to poorly constructed structures, slight damage to well-designed buildings</td>
</tr>
<tr>
<td>6.0</td>
<td>Destructive in populated areas</td>
</tr>
<tr>
<td>7.0</td>
<td>Serious damage over large geographic areas</td>
</tr>
<tr>
<td>8.0</td>
<td>Serious damage across areas of hundreds of miles</td>
</tr>
<tr>
<td>9.0</td>
<td>Serious damage across areas of hundreds of miles</td>
</tr>
<tr>
<td>10.0</td>
<td>Extremely rare, never recorded</td>
</tr>
</tbody>
</table>

**Typical Effects of Earthquakes of Various Magnitudes**

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>Very weak, no visible damage</td>
</tr>
<tr>
<td>2.0</td>
<td>Not felt by humans</td>
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<td>Often felt, usually no damage</td>
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<tr>
<td>4.0</td>
<td>Windows rattle, indoor items shake</td>
</tr>
<tr>
<td>5.0</td>
<td>Damage to poorly constructed structures, slight damage to well-designed buildings</td>
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<td>6.0</td>
<td>Destructive in populated areas</td>
</tr>
<tr>
<td>7.0</td>
<td>Serious damage over large geographic areas</td>
</tr>
<tr>
<td>8.0</td>
<td>Serious damage across areas of hundreds of miles</td>
</tr>
<tr>
<td>9.0</td>
<td>Serious damage across areas of hundreds of miles</td>
</tr>
<tr>
<td>10.0</td>
<td>Extremely rare, never recorded</td>
</tr>
</tbody>
</table>

**Common Core State Standards for Activity 22**

- **HSF-IF.B.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function $h(n)$ gives the number of person-hours it takes to assemble $n$ engines in a factory, then the positive integers would be an appropriate domain for the function.
- **HSF-IF.C.7** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.
- **HSF-IF.C.7e** Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

**Activity Standards Focus**

In Activity 22, students examine logarithmic functions and their graphs. They begin reviewing exponential functions. Then they examine the relationship between logarithmic and exponential functions and write equations using both forms. Students discover and use the properties of logarithms and graph logarithmic functions.

**Lesson 22-1**

**Pacing:** 1 class period

**Chunking the Lesson**

**TEACH**

**Bell-Ringer Activity**

Ask students to simplify each expression using properties of exponents.

1. $x^2 y^{-3} \cdot xy^2 = \frac{x^3}{y}$
2. $(2^3)^2 = 64$
3. $\frac{a^b}{a^{b-1}} = \frac{a}{1} = a$
4. $\frac{3^6}{3^2} = 19,683$

**Introduction, Summarizing, Paraphrasing, Debriefing**

The table of typical effects will help students understand the physical implications of different magnitudes. Lead a class discussion before starting this lesson to make certain that all students are comfortable with earthquake terminology.

Use the table on the next page to assess student understanding of magnitude and ground motion caused by an earthquake.

**Universal Access**

Have students look up all the meanings of the word magnitude to help them understand its meaning with respect to earthquakes. Discuss if and where they have seen or heard the word used before.
2. Complete the table to show how many times as great the ground motion is when caused by each earthquake as compared to a magnitude 0 earthquake.

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>Ground Motion Compared to Magnitude 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>10</td>
</tr>
<tr>
<td>2.0</td>
<td>100</td>
</tr>
<tr>
<td>3.0</td>
<td>1000</td>
</tr>
<tr>
<td>4.0</td>
<td>10,000</td>
</tr>
<tr>
<td>5.0</td>
<td>100,000</td>
</tr>
<tr>
<td>6.0</td>
<td>1,000,000</td>
</tr>
<tr>
<td>7.0</td>
<td>10,000,000</td>
</tr>
<tr>
<td>8.0</td>
<td>100,000,000</td>
</tr>
<tr>
<td>9.0</td>
<td>1,000,000,000</td>
</tr>
<tr>
<td>10.0</td>
<td>10,000,000,000</td>
</tr>
</tbody>
</table>

3. In parts a–c below, you will graph the data from Item 2. Let the horizontal axis represent the magnitude of the earthquake and the vertical axis represent the amount of ground motion caused by the earthquake as compared to a magnitude 0 earthquake.

a. Plot the data using a grid that displays $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. Explain why this grid is or is not a good choice.
   - A $[-10, 10] \times [-10, 10]$ window would not be appropriate since only the ordered pair (1, 10) would be plotted on the graph, as shown.

b. Plot the data using a grid that displays $-10 \leq x \leq 100$ and $-10 \leq y \leq 100$. Explain why this grid is or is not a good choice.
   - A $[-10, 100] \times [-10, 100]$ window would not be appropriate since only the ordered pairs (1, 10) and (2, 100) would be plotted, as shown. In addition, the x-axis does not need to be larger than 10 units.
Lesson 22-1
Exponential Data

c. Scales may be easier to choose if only a subset of the data is graphed and if different scales are used for the horizontal and vertical axes. Determine an appropriate subset of the data and a scale for the graph. Plot the data and label and scale the axes. Draw a function that fits the plotted data.

One possible answer is to choose the subset \{(1, 10), (2, 100), (3, 1000), (4, 10,000)\} and plot those points.

d. Write a function \(G(x)\) for the ground motion caused compared to a magnitude 0 earthquake by a magnitude \(x\) earthquake.

\[ G(x) = 10^x \]

Check Your Understanding

4. What is the domain of the function in Item 3d? Is the graph of the function continuous?

5. Use the graph from Item 3c to estimate how many times greater the ground motion of an earthquake of magnitude 3.5 is than a magnitude 0 earthquake. Solve the equation you wrote in Item 3d to check that your estimate is reasonable.

6. Make sense of problems. In Item 3, the data were plotted so that the ground motion caused by the earthquake was a function of the magnitude of the earthquake.

a. Is the ground motion a result of the magnitude of an earthquake, or is the magnitude of an earthquake the result of ground motion?

An earthquake’s magnitude is not assigned until an earthquake actually happens, so an earthquake’s magnitude is a result of ground motion.

b. Based your answer to part a, would you choose ground motion or magnitude as the independent variable of a function relating the two quantities? What would you choose as the dependent variable?

Ground motion should be the independent variable and magnitude should be the dependent variable of a function relating the two quantities.

Activity 22 • Logarithms and Their Properties
c. Make a new graph of the data plotted Item 3c so that the magnitude of the earthquake is a function of the ground motion caused by the earthquake. Scale the axes and draw a function that fits the plotted data.

7. Let the function you graphed in Item 6c be \( y = M(x) \), where \( M \) is the magnitude of an earthquake for which there is \( x \) times as much ground motion as a magnitude 0 earthquake.

a. Identify a reasonable domain and range of the function \( y = G(x) \) from Item 3d and the function \( y = M(x) \) in this situation. Use interval notation.

<table>
<thead>
<tr>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = G(x) )</td>
<td>[0, 10] ( \rightarrow ) (0, 10,000,000,000)</td>
</tr>
<tr>
<td>( y = M(x) )</td>
<td>(0, 10,000,000,000) ( \rightarrow ) [0, 10]</td>
</tr>
</tbody>
</table>

b. In terms of the problem situation, describe the meaning of an ordered pair on the graph of \( y = G(x) \) and \( y = M(x) \).

\( y = G(x) \)  Richter magnitude \( \rightarrow \) Ground motion compared to magnitude 0 earthquake

\( y = M(x) \)  Ground motion compared to magnitude 0 earthquake \( \rightarrow \) Richter magnitude
Lesson 22-1
Exponential Data

c. A portion of the graphs of \( y = G(x) \) and \( y = M(x) \) is shown on the same set of axes. Describe any patterns you observe.

![Graph of G(x) and M(x)]

Sample answer: The two functions are symmetric about the line \( y = x \). The values of \( x \) and \( y \) in \( M(x) \) are the values of \( y \) and \( x \) in \( G(x) \).

Check Your Understanding

8. How did you choose the scale of the graph you drew in Item 6c?
9. What is the relationship between the functions \( G \) and \( M \)?

LESSON 22-1 PRACTICE

How does the ground motion caused by earthquakes of these magnitudes compare?

10. Magnitude 5.0 compared to magnitude 2.0
11. Magnitude 7.0 compared to magnitude 0
12. Magnitude 6.0 compared to magnitude 5.0
13. A 1933 California earthquake had a Richter scale reading of 6.3. How many times more powerful was the Alaska 1964 earthquake with a reading of 8.3?
14. Critique the reasoning of others. Garrett said that the ground motion of an earthquake of magnitude 6 is twice the ground motion of an earthquake of magnitude 3. Is Garrett correct? Explain.

ACTIVITY 22 Continued

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand the functions are inverses. Have them list a few points from each function and reverse the coordinates to help them understand this.

Answers

8. reversed the axes from the graph for Item 3c
9. The functions \( G \) and \( M \) are inverse functions.

ASSESS

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 22-1 PRACTICE

10. Magnitude 5.0 is 1000 times greater than magnitude 2.0.
11. Magnitude 7.0 is 10,000,000 times greater than magnitude 0.
12. Magnitude 6.0 is 10 times greater than magnitude 5.0.
13. 10,000
14. No; the function does not increase linearly. The ground motion of an earthquake of magnitude 6 is 1000 times the ground motion of an earthquake of magnitude 3.

ADAPT

Check students’ answers to the Lesson Practice to ensure that they understand that the ground motion function is not linear. Have students refer back to the tables in Item 1 to confirm this.
Learning Targets:
- Use technology to graph \( y = \log x \).
- Evaluate a logarithm using technology.
- Rewrite exponential equations as their corresponding logarithmic equations.
- Rewrite logarithmic equations as their corresponding exponential equations.

**SUGGESTED LEARNING STRATEGIES:** Close Reading, Vocabulary Organizer, Create Representations, Quickwrite, Think-Pair-Share

The Richter scale uses a base 10 logarithmic scale. A base 10 logarithmic scale means that when the ground motion is expressed as a power of 10, the magnitude of the earthquake is the exponent. You have seen this function \( G(x) = 10^x \), where \( x \) is the magnitude, in Item 3d of the previous lesson. The function \( M \) is the inverse of an exponential function \( G \) whose base is 10. The algebraic rule for \( M \) is a common logarithmic function. Write this function as \( M(x) = \log x \), where \( x \) is the ground motion compared to a magnitude 0 earthquake.

1. Graph \( M(x) = \log x \) on a graphing calculator.
   a. Make a sketch of the calculator graph. Be certain to label and scale each axis.
   b. Use \( M \) to estimate the magnitude of an earthquake that causes 120,000 times the ground motion of a magnitude 0 earthquake. Describe what would happen if this earthquake were centered beneath a large city.
   c. Use \( M \) to determine the amount of ground motion caused by the 2002 magnitude 7.9 Denali earthquake compared to a magnitude 0 earthquake.

2. Complete the tables below to show the relationship between the exponential function base 10 and its inverse, the common logarithmic function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 10^x )</th>
<th>( y )</th>
<th>( \log y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>100</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1000</td>
<td>1000</td>
<td>3</td>
</tr>
<tr>
<td>( \log x )</td>
<td>( 10^x )</td>
<td>( y = \log x )</td>
<td>( \log y )</td>
</tr>
<tr>
<td>( 10^0 = 1 )</td>
<td>1</td>
<td>( \log 1 = 0 )</td>
<td></td>
</tr>
<tr>
<td>( 10^1 = 10 )</td>
<td>10</td>
<td>( \log (10) = 1 )</td>
<td></td>
</tr>
<tr>
<td>( 10^2 = 100 )</td>
<td>100</td>
<td>( \log (100) = 2 )</td>
<td></td>
</tr>
<tr>
<td>( 10^3 = 1000 )</td>
<td>1000</td>
<td>( \log (1000) = 3 )</td>
<td></td>
</tr>
</tbody>
</table>

The Common Logarithm Function

**TECHNOLOGY TIP**

The \( \log \) key on your calculator is for common, or base 10, logarithms.

**MATH TERMS**

A logarithm is an exponent to which a base is raised that results in a specified value.

A common logarithm is a base 10 logarithm, such as \( \log x \). \( \log 100 = 2 \) because \( 10^2 = 100 \).

**MATH TIP**

You can also write the equation \( y = \log x \) as \( x = 10^y \). In the equation \( y = \log x \), 10 is understood to be the base. Just as exponential functions can have bases other than 10, logarithmic functions can also be expressed with bases other than 10.
Lesson 22-2
The Common Logarithm Function

3. Use the information in Item 2 to write a logarithmic statement for each exponential statement.
   a. \(10^3 = 10,000\)
   b. \(10^{-1} = \frac{1}{10}\)
   \(\log(10,000) = 4\)
   \(\log\left(\frac{1}{10}\right) = -1\)

4. Use the information in Item 2 to write each logarithmic statement as an exponential statement.
   a. \(\log 100,000 = 5\)
   b. \(\log\left(\frac{1}{100}\right) = -2\)
   \(10^5 = 100,000\)
   \(10^{-2} = \frac{1}{100}\)

5. Evaluate each logarithmic expression without using a calculator.
   a. \(\log 1000\)
   b. \(\log\left(\frac{1}{10,000}\right) = -4\)

Check Your Understanding

6. What function has a graph that is symmetric to the graph of \(y = \log x\) about the line \(y = x\)?

7. Evaluate \(\log 10^x\) for \(x = 1, 2, 3,\) and 4.

8. Let \(f(x) = 10^x\) and \(g(x) = f^{-1}(x)\). What is the algebraic rule for \(g(x)\)?

LESSON 22-2 PRACTICE

9. Evaluate without using a calculator.
   a. \(\log 10^6\)
   b. \(\log 1,000,000\)
   c. \(\log\left(\frac{1}{100}\right)\)

10. Write an exponential statement for each.
    a. \(\log 10 = 1\)
    b. \(\log\left(\frac{1}{1,000,000}\right) = -6\)
    c. \(\log a = b\)

11. Write a logarithmic statement for each.
    a. \(10^5 = 10,000,000\)
    b. \(10^0 = 1\)
    c. \(10^n = n\)

12. Model with mathematics. The number of decibels \(D\) of a sound is modeled with the equation \(D = 10\log\left(\frac{I}{10^{-12}}\right)\) where \(I\) is the intensity of the sound measured in watts. Find the number of decibels in each of the following:
    a. whisper with \(I = 10^{-10}\)
    b. normal conversation with \(I = 10^{-6}\)
    c. vacuum cleaner with \(I = 10^{-4}\)
    d. front row of a rock concert with \(I = 10^{-1}\)
    e. military jet takeoff with \(I = 10^2\)

Activity 22 • Logarithms and Their Properties
My Notes

Properties of Logarithms

Learning Targets:

• Make conjectures about properties of logarithms.
• Write and apply the Product Property and Quotient Property of Logarithms.
• Rewrite logarithmic expressions by using properties.

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Create Representations, Look for a Pattern, Quickwrite, Guess and Check

You have already learned the properties of exponents. Logarithms also have properties.

1. Complete these three properties of exponents.
   \[a^m \cdot a^n = a^{m+n}\]
   \[\frac{a^m}{a^n} = a^{m-n}\]
   \[(a^m)^n = a^{mn}\]

2. Use appropriate tools strategically. Use a calculator to complete the tables below. Round each answer to the nearest thousandth.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.778</td>
</tr>
<tr>
<td>7</td>
<td>0.845</td>
</tr>
<tr>
<td>8</td>
<td>0.903</td>
</tr>
<tr>
<td>9</td>
<td>0.954</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

3. Add the logarithms from the tables in Item 2 to see if you can develop a property. Find each sum and round each answer to the nearest thousandth.

\[\log_2 2 + \log_3 3 = 0.778 = \log 6\]
\[\log_2 2 + \log_4 4 = 0.903 = \log 8\]
\[\log_2 2 + \log_5 5 = 1 = \log 10\]
\[\log_3 3 + \log_3 3 = 0.954 = \log 9\]

MINI-LESSON: Review of Exponent Properties

Students need to be familiar with the properties of exponents in this lesson. If students need to review these properties, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.
Lesson 22-3
Properties of Logarithms

4. Compare the answers in Item 3 to the tables of data in Item 2.
   a. Express regularity in repeated reasoning. Is there a pattern or property when these logarithms are added? If yes, explain the pattern that you have found.
   When two logarithms of like bases are added together, the result is the logarithm of the product of the input values.
   b. State the property of logarithms that you found by completing the following statement.
      \[ \log m + \log n = \log (mn) \]

5. Explain the connection between the property of logarithms stated in Item 4 and the corresponding property of exponents in Item 1.

6. Graph \( y_1 = \log 2 + \log x \) and \( y_2 = \log 2x \) on a graphing calculator.
   The two functions are identical because \( y_2 = \log (2x) = \log (2) + \log (x) = y_1 \) by the property in Item 4b.

Check Your Understanding

Identify each statement as true or false. Justify your answers.

7. \( \log mn = (\log m)(\log n) \)
   False; it is the sum of the logs: \( \log m + \log n \).

8. \( \log xy = \log x + \log y \)
   True by the property in Item 4b.

9. Make a conjecture about the property of logarithms that relates to the property of exponential equations that states the following:
   \( \frac{a^m}{a^n} = a^{n-m} \).
   The conjecture is \( \log (m) - \log (n) = \log \left( \frac{m}{n} \right) \).

10. Use the information from the tables in Item 2 to provide examples that support your conjecture in Item 9.
    Sample answers:
    \[ \log (4) - \log (2) = 0.301 = \log \left( \frac{4}{2} \right) = \log (2) \]
    \[ \log (6) - \log (2) = 0.477 = \log \left( \frac{6}{2} \right) = \log (3) \]

11. Graph \( y_1 = \log x - \log 2 \) and \( y_2 = \log \frac{x}{2} \) on a graphing calculator.
    What do you observe?
    The graphs of the two functions are identical.

ACTIVITY 22 Continued

3–4 (continued) Look for a Pattern, Debriefing
   Students should realize that when two logarithms with like bases are added, the result is the logarithm of the product of the input values.

5–6 Quickwrite, Create Representations, Debriefing
   Students may struggle with making the connection in Item 5. When exponential expressions with like bases are multiplied, the exponents are added. When logarithms with like bases are added, the result is the logarithm of the product of their inputs. Help students to see that logarithms are exponents that have been expressed using different notation.
   Students have looked at the property in Item 6 both numerically and analytically in terms of the corresponding property of exponents. Now they will validate the property graphically. A suggested graphing window would be \([0, 10]\) for the \(x\)-axis and \([-2, 2]\) for the \(y\)-axis.

Check Your Understanding
   Debrief students’ answers to these items to ensure that they understand the properties of logarithms presented in this lesson.

Answers
   7. False; it is the sum of the logs: \( \log m + \log n \).
   8. True by the property in Item 4b.

9–11 Guess and Check, Create Representations, Look for a Pattern, Debriefing
   Students will make a conjecture or guess in Item 9 and then validate or check it in the items that follow. Students who struggle with Item 9 should be directed to take a similar approach to what they did in Item 3 and Item 4. If students continue to have difficulty with this concept, have them answer Item 11 and then return to Item 9.
   In Items 10 and 11, students validate their conjecture both numerically and graphically. A suggested graphing window would be \([0, 10]\) for the \(x\)-axis and \([-2, 2]\) for the \(y\)-axis.
Check Your Understanding

Debrief students’ answers to these items to ensure that they can use the Product and Quotient Properties to rewrite logarithms.

Answers

12. Sample answers: \(\log(9 \cdot 4)\)
   \(= \log 9 + \log 4 = 0.954 + 0.602 = 1.556;\)
   \(\log(6 \cdot 6) = \log 6 + \log 6 = 0.778 + 0.778 = 1.556.\)

13. Sample answers: \(\log\left(\frac{10}{5}\right) = 1 - \log 5 = 1 - 0.699 = 0.301;\)
   \(\log 7 = 0.845; \log 3 = 0.477; \log 4 = 0.602\) (not 0.845).

14. Construct viable arguments. Show that \(\log(3 + 4) \neq \log 3 + \log 4.\)

Check Your Understanding

Use the information from the tables in Item 2 and the properties in Items 4b and 9.

12. Write two different logarithmic expressions to find a value for \(\log 36.\)
13. Write a logarithmic expression that contains a quotient and simplifies to 0.301.
14. Construct viable arguments. Show that \(\log(3 + 4) \neq \log 3 + \log 4.\)

LESSON 22-3 PRACTICE

Use the table of logarithmic values at the beginning of the lesson to evaluate the logarithms in Items 15 and 16. Do not use a calculator.

15. a. \(\log\left(\frac{8}{3}\right)\)
   \(= 0.903 - 0.477 = 0.426\)
   b. \(\log 24 = \log(8 \cdot 3) = \log 8 + \log 3 = 0.903 + 0.477 = 1.38\)
   c. \(\log 64 = \log(8 \cdot 8) = \log 8 + \log 8 = 0.903 + 0.903 = 1.806\)
   d. \(\log 27 = \log(3 \cdot 3 \cdot 3) = \log 3 + \log 3 + \log 3 = 0.477 + 0.477 + 0.477 = 1.431\)

16. a. \(\log\left(\frac{4}{9}\right) = \log 4 - \log 9\)
   \(= 0.602 - 0.954 = -0.352\)
   b. \(\log 2.25 = \log\left(\frac{9}{4}\right) = \log 9 - \log 4 = 0.954 - 0.602 = 0.352\)
   c. \(\log 144 = \log(4 \cdot 4 \cdot 9) = \log 4 + \log 4 + \log 9 = 0.602 + 0.602 + 0.954 = 2.158\)
   d. \(\log 81 = \log(9 \cdot 9) = \log 9 + \log 9 = 0.954 + 0.954 = 1.908\)

17. \(\log\left(\frac{2x}{3y}\right)\)
18. \(\log 8 + \log m - (\log 9 + \log n)\)
19. \(\log\left(\frac{8 + 2}{4}\right) = \log 4 = 0.602\)

ADAPT

Check students’ answers to the Lesson Practice to ensure that they understand how to evaluate a logarithmic expression and rewrite an expression as a single logarithm. As an additional activity, use index cards to create a game for students to match expressions. For example, \(\log\left(\frac{4}{9}\right)\) would match the expression \(\log 4 - \log 9.\)
Learning Targets:
- Make conjectures about properties of logarithms.
- Write and apply the Power Property of Logarithms.
- Identify each statement as true or false. Justify your answer.
- Use appropriate tools strategically.
- Express regularity in repeated reasoning.

My Notes

### Check Your Understanding

Identify each statement as true or false. Justify your answer.

1. 4. $2 \log \sqrt{m} = \log m$
2. 5. $\log 10^2 = \log 2^{10}$

6. Express regularity in repeated reasoning. The logarithmic properties that you conjectured and then verified in this lesson and the previous lesson are listed below. State each property.

   - **Product Property:** $\log m + \log n = \log (mn)$
   - **Quotient Property:** $\log (m) - \log (n) = \log \left(\frac{m}{n}\right)$
   - **Power Property:** $\log (m^n) = n \log (m)$

6–9 Think-Pair-Share, Create Representations, Debriefing

**Item 4**

Item 4 gives students an opportunity to reflect on the properties derived in the activity. After sharing answers with the entire group, make sure all responses are correct, students can record the information in their math notebooks.

Monitor group discussions to ensure that all members are participating. Pair or group students carefully to facilitate discussions and understanding of both routine language and mathematical terms.

**Check Your Understanding**

Debrief students’ answers to these items to ensure that they can correctly use properties of logarithms to justify their answers.

**Answers**

4. True; by the property in Item 1.
5. False; $\log 10^2 = 2$

### ACTIVITY 22 Continued

**Lesson 22-4**

**PLAN**

**Pacing:** 1 class period

**Chunking the Lesson**

- #1–3
- Check Your Understanding
- #6–9
- Check Your Understanding
- Lesson Practice

**TEACH**

**Bell-Ringer Activity**

Ask students to identify each statement as true or false. If the statement is false, have them correct it.

1. $\log 3 + \log 5 = \log 8$ [false; $\log 3 + \log 5 = \log 15$]
2. $\log 10 - \log 5 = \log 2$ [true]
3. $\log 1000 = 3$ [true]

1–3 Guess and Check, Create Representations, Look for a Pattern, Debriefing

Students may need coaching to develop the property in Item 1. There are two ways to approach this problem, following the patterns from Items 2–11. Students could try either $\log (u^3)$ or $\log (u^4)$. They may struggle with $\log (u^5)$ as there is no pattern to be found. However, if students try $\log (u^5)$, they will have more success if they recognize exponentiation as repeated multiplication and apply the Product Property of Logarithms:

$$\log (u^5) = \log (u \cdot u \cdot u \cdot u \cdot u)$$

In Item 2, students should apply the Product Property of Logarithms to verify their conjecture numerically. For example,

$$\log (3^2) = \log (3 \cdot 3) = \log 3 + \log 3 = 2 \log 3$$

In Item 3, students verify their conjecture graphically. A suggested graphing window would be $[0, 10]$ for the x-axis and $[-2, 2]$ for the y-axis.

**Check Your Understanding**

Debrief students’ answers to these items to ensure that they can correctly use properties of logarithms to justify their answers.

**Answers**

4. True; by the property in Item 1.
5. False; $\log 10^2 = 2$
More Properties of Logarithms

7. Use the properties from Item 6 to rewrite each expression as a single logarithm. Assume all variables are positive.
   a. \( \log x - \log 7 \)
   b. \( 2 \log x + \log y \)

8. Use the properties from Item 6 to expand each expression. Assume all variables are positive.
   a. \( \log 5xy^4 \)
   b. \( \log \frac{x}{y} \)

9. Rewrite each expression as a single logarithm. Then evaluate.
   a. \( \log 2 + \log 5 \)
   b. \( \log 5000 - \log 5 \)
   c. \( 2 \log 5 + \log 4 \)

10. Explain why \( \log(a + 10) \) does not equal \( \log a + 1 \).

11. Explain why \( \log(-100) \) is not defined.

ACTIVITY 22

Items 7–9 give students an opportunity to apply the logarithm properties. They also provide an opportunity to assess student understanding.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand that the log of a negative number is not a real number.

Answers

10. The log of a sum does not equal the sum of the logs; only the product equals the sum of the logs.
11. \( \log(-100) \) is not defined since it represents the exponent you would raise 10 to in order to get \(-100\). A base of 10 will never give a negative value.

ASSESS

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 22-4 PRACTICE

12. \( \log 100 = 2 \)
13. \( \log \frac{1}{10} = -1 \)
14. \( \log 10,000 = 4 \)
15. \( \log \frac{1}{100} = -2 \)
16. \( \log 100 \left( \frac{1}{100} \right) = \log 1 = 0 \)
17. \( \log b + 3 \log c + 2 \log d \)

ADAPT

Check students’ answers to the Lesson Practice to ensure that they can both rewrite logarithmic expressions as a single logarithm and expand a single logarithm into an expression. Students should easily and fluently go back and forth between the two forms. Allow students to use a calculator to check their work.

LESSON 22-4 PRACTICE

Attend to precision. Rewrite each expression as a single logarithm. Then evaluate the expression without using a calculator.

12. \( \log 5 + \log 20 \)
13. \( \log 3 - \log 30 \)
14. \( 2 \log 400 - \log 16 \)
15. \( \log \frac{1}{100} + 2 \log 2 \)
16. \( \log 100 + \log \left( \frac{1}{100} \right) \)
17. Expand the expression \( \log b^c d^e \).
**Logarithms and Their Properties**

**Earthquakes and Richter Scale**

**ACTIVITY 22 PRACTICE**

Write your answers on notebook paper. Show your work.

**Lesson 22-1**

[Graph showing Richter Magnitude vs. Ground Motion]

1. What is the y-intercept of the graph?
2. What is the x-intercept of the graph?
3. Is $M(x)$ an increasing or decreasing function?
4. Which of these statements are NOT true regarding the graph above?
   - A. The graph contains the point (1, 0).
   - B. The graph contains the point (10, 1).
   - C. The domain is $x > 0$.
   - D. The x-axis is an asymptote.

**Lesson 22-2**

5. Use a calculator to find a decimal approximation rounded to three decimal places.
   - a. $\log 47$
   - b. $\log 32.013$
   - c. $\log \left( \frac{5}{7} \right)$
   - d. $\log -20$

6. A logarithm is a(n)
   - A. variable.
   - B. constant.
   - C. exponent.
   - D. coefficient.

7. Write an exponential statement for each logarithmic statement below.
   - a. $\log 10,000 = 4$
   - b. $\log \frac{1}{1,000,000} = -9$
   - c. $\log a = 6$

8. Write a logarithmic statement for each exponential statement below.
   - a. $10^{-2} = \frac{1}{100}$
   - b. $10^3 = 10$
   - c. $10^4 = n$

9. Evaluate without using a calculator.
   - a. $\log 10^3$
   - b. $\log 100$
   - c. $\log \frac{1}{100,000}$

10. If $\log a = x$, and $10 < a < 100$, what values are acceptable for $x$?
    - A. $0 < x < 1$
    - B. $1 < x < 2$
    - C. $2 < x < 3$
    - D. $10 < x < 100$

**Lesson 22-3**

11. If $\log 2 = 0.301$ and $\log 3 = 0.447$, find each of the following using only these values and the properties of logarithms. Show your work.
    - a. $\log 6$
    - b. $\log \left( \frac{5}{7} \right)$
    - c. $\log 1.5$
    - d. $\log 18$

**ACTIVITY 22 Continued**

1. There is no y-intercept.
2. (1, 0)
3. $M(x)$ is increasing.
4. D
5. a. 1.672
   - b. 1.505
   - c. −0.146
   - d. not a real number
6. C
7. a. $10^4 = 10,000$
   - b. $10^{-9} = \frac{1}{1,000,000,000}$
   - c. $10^0 = 1$
8. a. $\log \frac{1}{100} = -2$
   - b. $\log 10 = 1$
   - c. $\log n = 4$
9. a. 5
   - b. 2
   - c. −5
10. B
11. a. $0.301 + 0.447 = 0.748$
    - b. $0.301 - 0.447 = -0.146$
    - c. $0.447 - 0.301 = 0.146$
    - d. $0.301 + 0.447 + 0.447 = 1.195$

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**ACTIVITY 22** Continued

12. B

13. Sample answer: Simplify

\[ \log 10^3 + \log 10^5 = \log 10^8 \]

Since \( \log 10 = 1 \), the log equation becomes \( 3 + 5 = 8 \). This is precisely the same operation used in the exponent product.

14. a. \( \log \frac{2x}{3y} \)

b. \( \log \frac{2}{3} \)

c. \( \log \frac{24-12}{6} = \log 48 \)

17. a. \( \log 3 + \log (8 + 8y) \)

b. \( \log (m + v) - \log 3 \)

c. \( \log 4 - \log (9 - u) \)

18. a. \( \log 3 + \log (8 + 8y) \)

b. \( \log (m + v) - \log 3 \)

c. \( \log 4 - \log (9 - u) \)

19. a. \( \log 3 \)

b. \( \log \frac{x}{y} \)

13. Explain the connection between the exponential equation \( (10^4 \cdot 10^5 = 10^9) \) and the logarithmic equation \( (\log 10^3 + \log 10^5 = \log 10^8) \).

14. Rewrite each expression as a single logarithm.

a. \( \log 2 + \log 3 = \log (2 \cdot 3) \)

b. \( \log 5 = \log 5 \)

c. \( (\log 24 + \log 12) - \log 6 \)

15. Expand each expression.

a. \( \log \frac{3x}{8y} \)

b. \( \log \frac{m + v}{3} \)

c. \( \log \frac{4}{y - u} \)

16. If \( \log 2 = 0.301 \) and \( \log 3 = 0.477 \), find each of the following using the properties of logarithms.

a. \( \log 4 \)

b. \( \log 27 \)

c. \( \log \sqrt[3]{2} \)

d. \( \log \sqrt[2]{12} \)

**Lesson 22-4**

17. Complete each statement to illustrate a property for logarithms.

a. Product Property \( \log uv = ? \)

b. Quotient Property \( \log \frac{v}{w} = ? \)

c. Power Property \( \log u^r = ? \)

18. Rewrite each expression as a single logarithm. Then evaluate without using a calculator.

a. \( \log 500 + \log 2 \)

b. \( 2 \log 3 + \log \frac{1}{7} \)

19. Expand each expression.

a. \( \log xy^2 \)

b. \( \log \frac{3x}{y} \)

c. \( \log a^b \)

20. If \( \log 8 = 0.903 \) and \( \log 3 = 0.477 \), find each of the following using the properties of logarithms.

a. \( \log 3^2 \)

b. \( \log (2 \cdot 3) \)

c. \( \log 8(3^2) \)

21. Write each expression without using exponents.

a. \( m \log n + \log n^m \)

b. \( \log (mn) \)

c. \( \log 2^x + \log 2^y \)

22. Which of the following statements is TRUE?

A. \( \log \frac{x}{y} = \log x \)

B. \( \log \frac{x}{y} = y \log x \)

C. \( \log (x + y) = \log x + \log y \)

D. \( \log \sqrt{x} = \frac{1}{2} \log x \)

**MATHEMATICAL PRACTICES**

**Reason Abstractly and Quantitatively**

23. Verify using the properties of logarithms that \( \log 10^x - \log 10^y = x - y \). Then evaluate for \( x = \pi \), using 3.14 for \( \pi \).
1. **Reason quantitatively.** Tell whether or not each table contains data that can be modeled by an exponential function. Provide an equation to show the relationship between x and y for the sets of data that are exponential.

   a. | x | 0 | 1 | 2 | 3 |
   --|---|---|---|---|
   y | 3 | 6 | 12 | 24 |

   b. | x | 0 | 1 | 2 | 3 |
   --|---|---|---|---|
   y | 2 | 4 | 6 | 8 |

   c. | x | 0 | 1 | 2 | 3 |
   --|---|---|---|---|
   y | 108 | 36 | 12 | 4 |

2. Tell whether or not each function is increasing. State whether or not the function is increasing or decreasing, and give the domain, range, and y-intercept of the function.

   a. \( y = 4 \left( \frac{2}{3} \right)^x \)  
   b. \( y = -3(4)^x \)

3. Let \( g(x) = 2(4)^{x-3} - 5 \).
   a. Describe the function as a transformation of \( f(x) = 4^x \).
   b. Graph the function using your knowledge of transformations.
   c. What is the horizontal asymptote of the graph of \( g(x) \)?

4. Rewrite each exponential equation as a common logarithmic equation.

   a. \( 10^3 = 1000 \)  
   b. \( 10^{-4} = \frac{1}{10,000} \)  
   c. \( 10^{-7} = 10,000,000 \)

5. **Make use of structure.** Rewrite each common logarithmic equation as an exponential equation.

   a. \( \log 100 = 2 \)  
   b. \( \log 100,000 = 5 \)  
   c. \( \log \frac{1}{10,000} = -5 \)

6. Evaluate each expression without using a calculator.

   a. \( \log 1000 \)  
   b. \( \log 1 \)  
   c. \( \log 2 + \log 50 \)

7. Evaluate using a calculator. Then rewrite each expression as a single logarithm without exponents and evaluate again as a check.

   a. \( \log 5 + \log 3 \)  
   b. \( \log 3^4 \)  
   c. \( \log 3 - \log 9 \)

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**Common Core State Standards for Embedded Assessment 2**

- **HSF-IF.B.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function \( h(n) \) gives the number of person-hours it takes to assemble \( n \) engines in a factory, then the positive integers would be an appropriate domain for the function.*

- **HSF-IF.B.6** Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

- **HSF-IF.C.7** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

- **HSF-IF.C.7e** Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

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**Embedded Assessment 2**

**Assessment Focus**

- Examining exponential patterns and functions
- Identifying and analyzing exponential graphs
- Transforming exponential functions
- Graphing and transforming natural base exponential functions
- Examining common logarithmic functions
- Understanding properties of logarithms

**Answer Key**

1. a. exponential; \( y = 3(2)^x \)  
   b. not exponential  
   c. exponential; \( y = 108 \left( \frac{1}{3} \right)^x \)

2. a. decreasing; domain: \((−\infty, \infty); \) range: \((0, \infty); \) y-intercept: 4  
   b. decreasing; domain: \((−\infty, 0); \) y-intercept: −3

3. To obtain the graph of \( g \), vertically stretch the graph of \( f \) by a factor of 2, horizontally translate to the left 3 units, and then vertically translate down 5 units.

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* For example, if the function \( h(n) \) gives the number of person-hours it takes to assemble \( n \) engines in a factory, then the positive integers would be an appropriate domain for the function.
Embedded Assessment 2

TEACHER TO TEACHER

You may wish to read through the scoring guide with students and discuss the differences in the expectations at each level. Check that students understand the terms used.

Unpacking Embedded Assessment 3

Once students have completed this Embedded Assessment, turn to Embedded Assessment 3 and unpack it with them. Use a graphic organizer to help students understand the concepts they will need to know to be successful on Embedded Assessment 3.

<table>
<thead>
<tr>
<th>Scoring Guide</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics Knowledge and Thinking (Items 1, 2, 3c, 4–7)</td>
<td>• Clear and accurate understanding of how to determine whether a table of data represents an exponential function</td>
<td>• Largely correct understanding of how to determine whether a table of data represents an exponential function</td>
<td>• Partial understanding of how to determine whether a table of data represents an exponential function</td>
<td>• Little or no understanding of how to determine whether a table of data represents an exponential function</td>
</tr>
<tr>
<td></td>
<td>• Clear and accurate understanding of the features of exponential functions and their graphs including domain and range</td>
<td>• Largely correct understanding of the features of exponential functions and their graphs including domain and range</td>
<td>• Partial understanding of the features of exponential functions and their graphs including domain and range</td>
<td>• Inaccurate or incomplete understanding of the features of exponential functions and their graphs including domain and range</td>
</tr>
<tr>
<td></td>
<td>• Fluency in evaluating and rewriting exponential and logarithmic equations and expressions</td>
<td>• Little difficulty when evaluating and rewriting exponential and logarithmic equations and expressions</td>
<td>• Some difficulty when evaluating and rewriting logarithmic and exponential equations and expressions</td>
<td>• Significant difficulty when evaluating and rewriting logarithmic and exponential equations and expressions</td>
</tr>
<tr>
<td>Problem Solving (Item 1)</td>
<td>• An appropriate and efficient strategy that results in a correct answer</td>
<td>• A strategy that may include unnecessary steps but results in a correct answer</td>
<td>• A strategy that results in some incorrect answers</td>
<td>• No clear strategy when solving problems</td>
</tr>
<tr>
<td>Mathematical Modeling / Representations (Items 1, 3b)</td>
<td>• Fluency in recognizing exponential data and modeling it with an equation</td>
<td>• Little difficulty in accurately recognizing exponential data and modeling it with an equation</td>
<td>• Some difficulty with recognizing exponential data and modeling it with an equation</td>
<td>• Significant difficulty with recognizing exponential data and modeling it with an equation</td>
</tr>
<tr>
<td></td>
<td>• Effective understanding of how to graph an exponential function using transformations</td>
<td>• Largely correct understanding of how to graph an exponential function using transformations</td>
<td>• Partial understanding of how to graph an exponential function using transformations</td>
<td>• Mostly inaccurate or incomplete understanding of how to graph an exponential function using transformations</td>
</tr>
<tr>
<td>Reasoning and Communication (Items 1a, 3a)</td>
<td>• Clear and accurate justification of whether or not data represented an exponential model</td>
<td>• Adequate justification of whether or not data represented an exponential model</td>
<td>• Misleading or confusing justification of whether or not data represented an exponential model</td>
<td>• Incomplete or inadequate justification of whether or not data represented an exponential model</td>
</tr>
<tr>
<td></td>
<td>• Precise use of appropriate math terms and language to describe a function as a transformation of another function</td>
<td>• Adequate and correct description of a function as a transformation of another function</td>
<td>• Misleading or confusing description of a function as a transformation of another function</td>
<td>• Incomplete or mostly inaccurate description of a function as a transformation of another function</td>
</tr>
</tbody>
</table>

Common Core State Standards for Embedded Assessment 2 (cont.)

HSF-BF.A.1 Write a function that describes a relationship between two quantities.*

HSF-BF.A.1a Determine an explicit expression, a recursive process, or steps for calculation from a context.

HSF-BF.B.3 Identify the effect on the graph of replacing \( f(x) \) by \( f(x) + k \), \( kf(x) \), \( f(kx) \), and \( f(x + k) \) for specific values of \( k \) (both positive and negative); find the value of \( k \) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
Inverse Functions: Exponential and Logarithmic Functions

Learning Targets:
1. Use composition to verify two functions as inverse.
2. Define the logarithm of y with base b.
3. Write the Inverse Properties for logarithms.

SUGGESTED LEARNING STRATEGIES: Close Reading, Create Representations

In the first unit, you studied inverses of linear functions. Recall that two functions f and g are inverses of each other if and only if f(g(x)) = x for all x in the domain of g, and g(f(x)) = x for all x in the domain of f.

1. Find the inverse function g(x) of the function f(x) = 2x + 1. Show your work.
   - y = 2x + 1
   - x = 2y + 1
   - y = \frac{x - 1}{2}
   - g(x) = \frac{x - 1}{2}

2. Use the definition of inverse functions to prove that f(x) = 2x + 1 and the g(x) function you found in Item 1 are inverse functions.
   - f(g(x)) = 2\left(\frac{x - 1}{2}\right) + 1 = x - 1 + 1 = x
   - g(f(x)) = \frac{(2x + 1) - 1}{2} = \frac{2x}{2} = x

3. Graph f(x) = 2x + 1 and its inverse g(x) on the grid below. What is the line of symmetry between the graphs?
   - The line of symmetry is y = x.

In a previous activity, you investigated exponential functions with a base of 10 and their inverse functions, the common logarithmic functions. Recall in the Richter scale situation that G(x) = 10^x, where x is the magnitude of an earthquake. The inverse function is M(x) = log x, where x is the ground motion compared to a magnitude 0 earthquake.

**MATH TIP**
To find the inverse of a function algebraically, interchange the x and y variables and then solve for y.

**Activity Standards Focus**
In Activity 23, students extend the concept of logarithms to bases other than 10. They also extend their knowledge of inverse functions to include the inverse relationship between y = b^x and y = log_b x. Students will discover and apply properties of logarithms and apply the concept of graphing by transformations to logarithmic functions.

**Lesson 23-1**

**Pacing:** 1 class period

**Chunking the Lesson**
#1–3  #4  #5–6  #7–8

Check Your Understanding

**Lesson Practice**

**TEACH**

**Bell-Ringer Activity**
Present the following graph of f(x), and ask students to sketch the graphs of f(x) and g(x), the reflection of f(x), over the line y = x. Have students use the graph to find g(-1), g(2), and g(5).

[g(-1) = 4, g(2) = 2, g(5) = 0]

1–3 Activating Prior Knowledge, Create Representations These items review prior work with finding the inverse of a linear function and then graphing a function and its inverse. This then becomes the foundation for working with inverses of logarithmic functions.

**Common Core State Standards for Activity 23**

- HSF-BF.A.1 Write a function that describes a relationship between two quantities.*
- HSF-BF.A.1c (+) Compose functions.
- HSF-BF.B.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
- HSF-BF.B.4 Find inverse functions.
- HSF-BF.B.4b (+) Verify by composition that one function is the inverse of another.
4. A part of each of the graphs of \( y = G(x) \) and \( y = M(x) \) is shown below. What is the line of symmetry between the graphs? How does that line compare with the line of symmetry in Item 3?

The line of symmetry is \( y = x \). It is the same as the line of symmetry in Item 3.

Logarithms with bases other than 10 have the same properties as common logarithms.

The logarithm of \( y \) with base \( b \), where \( y > 0, b > 0, b \neq 1 \), is defined as:

\[
\log_b y = x \quad \text{if and only if} \quad y = b^x.
\]

The exponential function \( y = b^x \) and the logarithmic function \( y = \log_b x \), where \( b > 0 \) and \( b \neq 1 \), are inverse functions.

5. Let \( g(x) = f^{-1}(x) \), the inverse of function \( f \). Write the rule for \( g \) for each function \( f \) given below.

\[ a. \quad f(x) = 5^x \quad b. \quad f(x) = \log_4 x \quad c. \quad f(x) = \log_e x \]

\[ g(x) = \log_5 x \quad g(x) = \log_4 x \quad g(x) = \log_e x \]

Logarithms with base \( e \) are called natural logarithms, and “\( \log_e \)” is written \( \ln \). So, \( \log_e x \) is written \( \ln x \).

6. Use the functions from Item 5. Complete the expression for each composition.

\[ a. \quad f(x) = 5^x \]

\[ f(g(x)) = 5^{\log_5 x} = x \]

\[ g(f(x)) = \log_5 5^x = x \]

\[ b. \quad f(x) = \log_4 x \]

\[ f(g(x)) = \log_4 4^x = x \]

\[ g(f(x)) = 4^{\log_4 x} = x \]

\[ c. \quad f(x) = e^x \]

\[ f(g(x)) = e^{\log_e x} = x \]

\[ g(f(x)) = \ln e^x = x \]
Lesson 23-1

Logarithms in Other Bases

7. Use what you learned in Item 6 to complete these Inverse Properties of Logarithms. Assume \( b > 0 \) and \( b \neq 1 \).
   \( a. \) \( b^x \cdot b^y = \) \( x \) \( b. \) \( \log_b b^x = \) \( x \)

8. Simplify each expression.
   \( a. \) \( 6^{\log_6 x} \) \( b. \) \( \log_3 3^x \)
   \( c. \) \( \log_{10} 10^{x} \) \( d. \) \( \log_b x \)
   \( e. \) \( \ln e^x \) \( f. \) \( \log_b x \)

Check Your Understanding

9. Describe the process you use to find the inverse function \( g(x) \) if \( f(x) = 7x + 8 \).
10. Construct viable arguments. Look at the graphs in Items 3 and 4. What can you conclude about the line of symmetry for a function and its inverse?
11. Answer each of the following as true or false. If false, explain your reasoning.
   \( a. \) The “\(-1\)” function notation \( f^{-1} \) means \( \frac{1}{f} \).
   \( b. \) Exponential functions are the inverse of logarithmic functions.
   \( c. \) If the inverse is a function, then the original must be a function.

LESSON 23-1 PRACTICE

Let \( g(x) = f^{-1}(x) \), the inverse of function \( f \). Write the rule for \( g \) for each function \( f \) given below.

12. \( f(x) = 3x - 8 \)
13. \( f(x) = \frac{1}{2}x + 5 \)
14. \( f(x) = 5x - 6 \)
15. \( f(x) = -x + 7 \)
16. \( f(x) = 7^x \)
17. \( f(x) = e^x \)
18. \( f(x) = \log_{12} x \)
19. \( f(x) = \ln x \)

Simplify each expression.
20. \( \log_6 9^x \)
21. \( 15^{\log_{15} x} \)
22. \( \ln e^x \)
23. \( \log_{10} x \)

Answers

9. Interchange \( x \) and \( y \) in the original function, and then solve for \( y \).
   \( x = 7y + 8, x - 8 = 7y, y = \frac{x - 8}{7} \) or \( f^{-1}(x) = \frac{x - 8}{7} \)
10. The line of symmetry for a function and its inverse is \( x = y \).
11. \( a. \) False; it means the inverse of function \( f \).
    \( b. \) true
    \( c. \) False; the original is not necessarily a function.
### Lesson 23-2

#### Properties of Logarithms and the Change of Base Formula

#### Learning Targets:
- Apply the properties of logarithms in any base.
- Compare and expand logarithmic expressions.
- Use the Change of Base Formula.

#### SUGGESTED LEARNING STRATEGIES:
Create Representations, Close Reading

When rewriting expressions in exponential and logarithmic form, it is helpful to remember that a *logarithm is an exponent*. The exponential statement $2^3 = 8$ is equivalent to the logarithmic statement $\log_2 8 = 3$.

1. Express each exponential statement as a logarithmic statement.
   - a. $3^4 = 81$  \( \log_3 81 = 4 \)
   - b. $6^{-2} = \frac{1}{36}$  \( \log_6 \left( \frac{1}{36} \right) = -2 \)
   - c. $e^0 = 1$  \( \ln 1 = 0 \)

2. Express each logarithmic statement as an exponential statement.
   - a. $\log_3 16 = 2$  \( 3^2 = 9 \)
   - b. $\log_4 125 = 3$  \( 4^3 = 64 \)
   - c. $\ln 1 = 0$  \( e^0 = 1 \)

3. Evaluate each expression without using a calculator.
   - a. $\log_2 32$  \( 5 \)
   - b. $\log_4 \left( \frac{1}{64} \right)$  \( -3 \)
   - c. $\log_7 3$  \( 3 \)
   - d. $\log_{12} 1$  \( 0 \)

4. Why is the value of $\log_{-2} 16$ undefined?
5. Critique the reasoning of others. Mike said that the $\log_3$ of $\frac{1}{27}$ is undefined, because $3^{-3} = \frac{1}{27}$ and a log cannot have a negative value. Is Mike right? Why or why not?

The Product, Quotient, and Power Properties of common logarithms also extend to bases other than base 10.

6. Use the given property to rewrite each expression as a single logarithm. Then evaluate each logarithm in the equation to see that both sides of the equation are equal.
   - a. Product Property:  \( \log_2 4 + \log_2 8 = \log_2 (4 \cdot 8) = \log_2 32 = 5 \)
   - b. Quotient Property:  \( \log_3 27 - \log_3 3 = \log_3 \left( \frac{27}{3} \right) = \log_3 9 = 2 \)
   - c. Power Property:  \( 2 \log_2 5 = \log_2 5^2 = \log_2 25 = 2.2 \)

### Check Your Understanding

4. The value of $b$ must be greater than 0, so $\log_{-2} 16$ is undefined.
5. Mike is mistaken; a logarithm can have a negative value, but it cannot have a negative base.
Lesson 23-2
Properties of Logarithms and the Change of Base Formula

7. Expand each expression. Assume all variables are positive.
   a. \(\log_7 \frac{x}{y} = \log_7 x - \log_7 y\)
   b. \(\log_4 x^2y = 2 \log_4 x + \log_4 y\)
   c. \(\ln \left( \frac{x}{y} \right) = 2 \ln x - 3 \ln y\)

8. Assume that \(x\) is any real number, and decide whether the statement is always true, sometimes true, or never true. If the statement is sometimes true, give the conditions for which it is true.
   a. \(\log 7 \log 5 \log 7 \log 5 = \neq\) never true
   b. \(\log_5 5 = \) always true
   c. \(2^{\log_5 x^2} = x^2\) sometimes true, when \(x \neq 0\)
   d. \(\log_3 3 + \log_4 4 - \log_4 x = \log_4 15\) sometimes true, when \(x = 1\)
   e. \(2 \ln x = \ln x + \ln x\) sometimes true, when \(x > 0\)

Check Your Understanding

9. Attend to precision. Why is it important to specify the value of the variables as positive when using the Product, Quotient, and Power Properties of logarithms? Use Item 7 to state an example.

10. Simplify the following expression:
    \(\log 7 - \log 5\)

Sometimes it is useful to change the base of a logarithmic expression. For example, the log key on a calculator is for common, or base 10, logs. Changing the base of a logarithm to 10 makes it easier to work with logarithms on a calculator.

11. Use the common logarithm function on a calculator to find the numerical value of each expression. Write the value in the first column of the table. Then write the numerical value using logarithms in base 2 in the second column.

<table>
<thead>
<tr>
<th>Numerical Value</th>
<th>(\log_2 \sigma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\log 2)</td>
<td>(\log_2 1)</td>
</tr>
<tr>
<td>(\log 2)</td>
<td>(\log_2 2)</td>
</tr>
<tr>
<td>(\log 4)</td>
<td>(\log_2 2)</td>
</tr>
<tr>
<td>(\log 8)</td>
<td>(\log_2 3)</td>
</tr>
<tr>
<td>(\log 16)</td>
<td>(\log_2 4)</td>
</tr>
<tr>
<td>(\log 2^N)</td>
<td>(\log_2 N)</td>
</tr>
</tbody>
</table>

ACTIVITY 23

7–8 Quickwrite, Debriefing

In Item 7, students apply the Product, Quotient, and Power Properties to expand logarithmic expressions.

In Item 8a, point out that \(\log 7 \log 5 \log 7 \log 5 = \neq\) never true. Students may mistakenly believe that this statement is always true, rather than never true. Item 8b is an identity that holds true for all \(x\). Item 8c is sometimes true. It is true for all real numbers except 0. When \(x = 0\), the left side of the equation is undefined and the right side is equal to 0. Item 8d is sometimes true. It is only true when \(x = 1\), and therefore \(\log x = 0\). Part e is sometimes true. It is only true when \(x\) is greater than 0.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand the Product, Quotient, and Power Properties for logarithms with bases other than 10. Ask students to write these properties using natural logarithms.

Answers

9. Suppose \(y < 0\), in \(\log_4 x^2y\). Then \(x^2y\) would be negative, and the logarithm of a negative number is undefined because \(b > 0\).

10. \(\log 2\)

11 Create Representations, Note Taking, Look for a Pattern

Students can use a calculator to evaluate common logarithmic expressions of the form \(\log (N) \div \log (2)\), and then look for a pattern to find an equivalent expression of the form \(\log_2 N\).
ACTIVITY 23 Continued

12 Create Representations, Note Taking, Look for a Pattern  Students generalize the pattern from base 2 to base b. Students should add to their math journals the Change of Base Formula: \( \log_b x = \frac{\log_a x}{\log_a b} \), where \( b, x > 0 \), \( b \neq 1 \).

13 Debriefing  Students use the Change of Base Formula to approximate \( \log_2 12 \) on a calculator, first mentally approximating the value and then checking with a calculator to see that their answer is reasonable.

Differentiating Instruction

Extend students’ knowledge of the Change of Base Formula. Item 12 shows the Change of Base Formula written in terms of the common logarithm because students can evaluate common logarithms on a calculator. However, this property of logarithms can be generalized to any base \( b \): \( \log_b a = \frac{\log_c a}{\log_c b} \), where \( a, b, c > 0 \) and \( b, c \neq 0 \). Challenge students to use the Change of Base Formula with a base other than 10 to evaluate the following logarithms without a calculator.

- a. \( \log_{10} 8 \) (base change to base 2)
- b. \( \log_{27} 3 \) (base change to base 3)

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand the Change of Base Formula. Ask students to rewrite the Change of Base Formula using natural logarithms.

ASSESS

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students’ answers to the Lesson Practice to ensure that they understand how to rewrite exponential and logarithmic expressions in an equivalent form. As needed, have students practice creating equivalent expressions by writing each of the elements of an expression on a different index card or piece of paper. Then have students reposition the cards to form equivalent expressions.

Answers

- a. \( \frac{\log 32}{\log 5} \approx 2.153 \)
- b. \( \frac{\log 104}{\log 3} \approx 4.228 \)
- 15. Sample answer: 12 lies between two powers of 2, 8 and 16. \( \log_{10} 8 = 3; \) \( \log_{10} 16 = 4 \). Therefore, \( \log_{10} 12 \) lies between 3 and 4.

Check Your Understanding

14. Change each expression to a logarithmic expression in base 10. Use a calculator to find the value to three decimal places.

- a. \( \log(32) \)
- b. \( \log(104) \)

15. In Item 13, how do you find out which values the value of \( \log_2 12 \) lies between?

LESSON 23-2 PRACTICE

Write a logarithmic statement for each exponential statement.

- 16. \( 7^3 = 343 \)
- 17. \( 3^7 = \frac{1}{9} \)
- 18. \( e^u = u \)

Write an exponential statement for each logarithmic statement.

- 19. \( \log_{18} 5.64 = 4 \)
- 20. \( \log_{1296} 0.43 \approx 0.43 \)

21. Evaluate each expression without using a calculator.

- a. \( \log_{18} 5.64 \)
- b. \( \log_{25} 4 \)
- c. \( \log_{42} 32 \)

22. Use a calculator to find the value of \( \log_2 4.170 \) to three decimal places.

23. Change each expression to a logarithmic expression in base 10. Use a calculator to find the value to three decimal places.

- a. \( \log_7 2 \)
- b. \( \log_{18} 5.64 \)
- c. \( \log_{42} 32 \)
Learning Targets:
- Find intercepts and asymptotes of logarithmic functions.
- Determine the domain and range of a logarithmic function.
- Write and graph transformations of logarithmic functions.

Suggested Learning Strategies: Create Representations, Look for a Pattern, Close Reading, Quickwrite

1. Examine the function \( f(x) = 2^x \) and its inverse, \( g(x) = \log_2 x \).
   a. Complete the table of data for \( f(x) = 2^x \). Then use that data to complete a table of values for \( g(x) = \log_2 x \).
   
   \[
   \begin{array}{c|c|c}
   x & f(x) = 2^x & x & g(x) = \log_2 x \\
   \hline
   -2 & \frac{1}{4} & -2 & \frac{1}{2} \\
   -1 & \frac{1}{2} & -1 & \frac{1}{4} \\
   0 & 1 & 0 & 1 \\
   1 & 2 & 1 & 0 \\
   2 & 4 & 2 & 1 \\
   \end{array}
   \]
   b. Graph both \( f(x) = 2^x \) and \( g(x) = \log_2 x \) on the same grid.
   c. What are the \( x \)- and \( y \)-intercepts for \( f(x) = 2^x \) and \( g(x) = \log_2 x \)?
      For \( f(x) = 2^x \), the \( y \)-intercept is 1, and there is no \( x \)-intercept.
      For \( g(x) = \log_2 x \), the \( x \)-intercept is 1, and there is no \( y \)-intercept.
   d. What is the line of symmetry between the graphs of \( f(x) = 2^x \) and \( g(x) = \log_2 x \)? \( y = x \)
   e. State the domain and range of each function using interval notation.
      \( f(x) = 2^x \) \( \text{Domain} \ (\neg \infty, \infty) \text{ Range} \ (0, \infty) \)
      \( g(x) = \log_2 x \) \( \text{Domain} \ (0, \infty) \text{ Range} \ (\neg \infty, \infty) \)
   f. What is the end behavior of the graph of \( f(x) = 2^x \)?
      As \( x \) approaches \( \neg \infty \), \( y \) approaches 0. As \( x \) approaches \( \infty \), \( y \) approaches \( \infty \).
   g. What is the end behavior of the graph of \( g(x) = \log_2 x \)?
      As \( x \) approaches \( \neg \infty \), \( y \) approaches \( -\infty \). As \( x \) approaches \( \infty \), \( y \) approaches \( \infty \).
**ACTIVITY 23** Continued

**2 Look for a Pattern, Think-Pair-Share, Debriefing** Students perform an investigation similar to the work they did in Item 1, with base $e$ instead of base 2. Have students discuss the similarities and differences in the answers to Items 1 and 2.

**ELL Support**
To support students’ language acquisition, monitor their listening skills and understanding as they compare Items 1 and 2. Carefully pair students to ensure that all students participate in and learn from the discussion.

**Technology Tip**
If students do not have a natural logarithm key on their calculators, remind them that $\ln x = \frac{\log x}{\log e}$.
Students can use the common logarithm key and a decimal approximation of $e$ as an alternate method of evaluating natural logarithms.

For additional technology resources, visit SpringBoard Digital.

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**Lesson 23–3 Graphs of Logarithmic Functions**

**h.** Write the equation of any asymptotes of each function.

- For $f(x) = 2^x$: $y = 0$
- For $g(x) = \log_2 x$: $x = 0$

**2.** Examine the function $f(x) = e^x$ and its inverse, $g(x) = \ln x$.

**a.** Complete the table of data for $f(x) = e^x$. Then use those data to complete a table of values for $g(x) = \ln x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$e^x$</th>
<th>$x$</th>
<th>$\ln x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2</td>
<td>0.135</td>
<td>0.135</td>
<td>−2</td>
</tr>
<tr>
<td>−1</td>
<td>0.368</td>
<td>0.368</td>
<td>−1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2.718</td>
<td>2.718</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>7.390</td>
<td>7.390</td>
<td>2</td>
</tr>
</tbody>
</table>

**b.** Graph both $f(x) = e^x$ and $g(x) = \ln x$ on the same grid.

**c.** What are the $x$- and $y$-intercepts for $f(x) = e^x$ and $g(x) = \ln x$?
- For $f(x) = e^x$, the $y$-intercept is 1, and there is no $x$-intercept.
- For $g(x) = \ln x$, the $x$-intercept is 1, and there is no $y$-intercept.

**d.** What is the line of symmetry between the graphs of $f(x) = e^x$ and $g(x) = \ln x$? $y = x$

**e.** State the domain and range of each function using interval notation.

- For $f(x) = e^x$: Domain $(-\infty, \infty)$, Range $(0, \infty)$
- For $g(x) = \ln x$: Domain $(0, \infty)$, Range $(-\infty, \infty)$

**f.** What is the end behavior of the graph of $f(x) = e^x$?
- As $x$ approaches $-\infty$, $y$ approaches 0.
- As $x$ approaches $\infty$, $y$ approaches $\infty$.

**g.** What is the end behavior of the graph of $g(x) = \ln x$?
- As $x$ approaches 0, $y$ approaches $-\infty$.
- As $x$ approaches $\infty$, $y$ approaches $\infty$.

**h.** Write the equation of any asymptotes of each function.

- For $f(x) = e^x$: $y = 0$
- For $g(x) = \ln x$: $x = 0$
Lesson 23-3
Graphs of Logarithmic Functions

Check Your Understanding

3. Make sense of problems. From the graphs you drew for Items 1 and 2, draw conclusions about the behavior of inverse functions with respect to:
   a. the intercepts
   b. the end behavior
   c. the asymptotes

4. If a function has an intercept of (0, 0), what point, if any, will be an intercept for the inverse function?

Transformations of the graph of the function \( f(x) = \log_b x \) can be used to graph functions of the form \( g(x) = a \log_b (x - c) + d \), where \( b > 0, b \neq 1 \). You can draw a quick sketch of each parent graph, \( f(x) = \log_b x \), by plotting the points \( \left( \frac{1}{B}, -1 \right) \), \( (1, 0) \), and \( (B, 1) \).

5. Sketch the parent graph \( f(x) = \log_b x \) on the axes below. Then, for each transformation of \( f \), provide a verbal description and sketch the graph, including asymptotes.
   a. \( g(x) = 3 \log_2 x \)
   b. \( h(x) = 3 \log_2 (x + 4) \)
   c. \( j(x) = 3 \log_2 (x + 4) - 2 \)
   d. \( k(x) = \log_2 (8x) \)

6. Explain how the function \( j(x) = 3 \log_2 (x + 4) - 2 \) can be entered on a graphing calculator using the common logarithm key. Then graph the function on a calculator and compare the graph to your answer in Item 5c.

\[
3 \log_2 (x + 4) - 2 \text{ as } j(x) = \frac{3 \log_2 (x + 4) - 2}{\log_2 (2)}
\]

ACTIVITY 23 Continued

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand the behavior of the graphs of inverse exponential and logarithmic functions. Ask students to explain the relationship between the domains and ranges of inverse functions and how this relationship supports the answers to Item 3.

Answers

3. a. If the function has a \( y \)-intercept of 1, then the inverse function has an \( x \)-intercept of 1.
   b. If the function approaches 0 on the \( y \)-axis as \( x \) approaches infinity, then in the inverse function, \( y \) approaches infinity as \( x \) approaches 0.
   c. If the function has an asymptote at \( x = 0 \), then the inverse has an asymptote at \( y = 0 \).

4. \((0, 0)\)

5 Activating Prior Knowledge, Create Representations, Quickwrite, Debriefing

The Math Tip shows students why they can get a quick sketch of the graph of \( g(x) = \log_b x \) by plotting the points \( \left( \frac{1}{B}, -1 \right), (1, 0), \) and \( (B, 1) \).

Students are expected to use their prior knowledge of graphing other functions by transformations to sketch transformations of the logarithmic function.

6 Quickwrite

Students can use the Change of Base Formula in order to graph the function on a calculator.
7 Explain how the parameters $a$, $c$, and $d$ transform the parent graph $f(x) = \log_b x$ to produce a graph of the function $g(x) = a \log_b (x - c) + d$. Transformations of the parent graph $f(x) = \log_b x$ that produce the graph of $g(x) = a \log_b (x - c) + d$ are described below.

$|a| > 1 \Rightarrow$ a vertical stretch by a factor of $a$

$0 < |a| < 1 \Rightarrow$ a vertical shrink by a factor of $a$

$a < 0 \Rightarrow$ a reflection over the $x$-axis

$c > 0 \Rightarrow$ a horizontal translation right $c$ units

$c < 0 \Rightarrow$ a horizontal translation left $c$ units

$d > 0 \Rightarrow$ a vertical shift down $d$ units

$d < 0 \Rightarrow$ a vertical shift up $d$ units

Check Your Understanding

8. Look for and make use of structure.
   a. Compare the effect of $a$ in a logarithmic function $a \log_b x$ to a quadratic function $ax^2$ (assume $a$ is positive).
   b. Compare the effect of $c$ in a logarithmic function $\log_b (x - c)$ to $c$ in a quadratic function $(x - c)^2$.

Lesson 23-3 Practice

9. Given an exponential function that has a $y$-intercept of 1 and no $x$-intercept, what is true about the intercepts of the function’s inverse?

10. Make sense of problems. The inverse of a function has a domain of $(-\infty, \infty)$ and a range of $(0, \infty)$. What is true about the original function’s domain and range?

Model with mathematics. Graph each function, using a parent graph and the appropriate transformations. Describe the transformations.

11. $f(x) = 2 \log_3 (x) - 6$
12. $f(x) = \log_2 (x - 5) + 1$
13. $f(x) = \frac{1}{2} \log_4 x$
14. $f(x) = \log_2 (x + 4) - 3$

Check students’ answers to the Lesson Practice to ensure that they understand graphing logarithmic and exponential functions. As needed, provide students with additional practice using transformations to graph functions. Students may benefit from additional practice graphing quadratic functions or square root functions to better understand the effect of the parameters on the graph of the parent function.
In the figure, the graphs of $f(x)$ and $g(x)$ are shown, where $g(x) = f^{-1}(x)$. The given functions are $f(x) = 3x - 2$ and $g(x) = x^2 - 4x + 1$.

To find $g(f(x))$, we substitute $f(x)$ into $g(x)$:

$$g(f(x)) = (3x - 2)^2 - 4(3x - 2) + 1$$

Expanding and simplifying:

$$= 9x^2 - 12x + 4 - 12x + 8 + 1$$

$$= 9x^2 - 24x + 13$$

To find $f(g(x))$, we substitute $g(x)$ into $f(x)$:

$$f(g(x)) = 3(x^2 - 4x + 1) - 2$$

Expanding and simplifying:

$$= 3x^2 - 12x + 3 - 2$$

$$= 3x^2 - 12x + 1$$

Therefore, $g(f(x)) = 9x^2 - 24x + 13$ and $f(g(x)) = 3x^2 - 12x + 1$. The graphs of $g(f(x))$ and $f(g(x))$ are also shown in the figure.
Rewrite each expression as a single, simplified logarithmic term. Assume all variables are positive.

30. \(\log_2 32 + \log_2 2\)
31. \(\log_3 x^2 - \log_3 y\)
32. \(\ln x + \ln 2\)
33. \(3 \ln x\)

Evaluate each expression without using a calculator.

34. \(\log_{12} 12\)
35. \(\log_{7} 343\)
36. \(\log_{7} 49\)
37. \(\log_{3} 81\)

Change each expression to a logarithmic expression in base 10. Use a calculator to find the value to three decimal places.

38. \(\log_{4} 20\)
39. \(\log_{20} 4\)
40. \(\log_{5} 45\)
41. \(\log_{3} 18\)

Lesson 23-3

42. If the domain of a logarithmic function is \((0, \infty)\) and the range is \((-\infty, \infty)\), what are the domain and range of the inverse of the function?
   A. domain: \((-\infty, \infty)\), range: \((-\infty, \infty)\)
   B. domain: \((0, \infty)\), range: \((-\infty, \infty)\)
   C. domain: \((-\infty, \infty)\), range: \((-\infty, \infty)\)
   D. domain: \((-\infty, \infty)\), range: \((0, \infty)\)

Graph each function, using a parent graph and the appropriate transformations. Describe the transformations.

43. \(f(x) = 3 \log_2 (x) - 1\)
44. \(f(x) = \log_3 (x - 4) + 2\)
45. \(f(x) = \frac{1}{2} \log_4 x\)
46. \(f(x) = \log_2 (x + 3) - 4\)

MATHEMATICAL PRACTICES

Model with Mathematics

47. Given the function \(f(x) = 2^x + 1\)
   a. Give the domain, range, \(y\)-intercept, and any asymptotes for \(f(x)\). Explain.
   b. Draw a sketch of the graph of the function on a grid. Describe the behavior of the function as \(x\) approaches \(\infty\) and as \(x\) approaches \(-\infty\).

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.
ACTIVITY 24

Logarithmic and Exponential Equations and Inequalities

College Costs
Lesson 24-1 Exponential Equations

Learning Targets:
* Write exponential equations to represent situations.
* Solve exponential equations.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Create Representations, Vocabulary Organizer, Note Taking, Group Presentation

Wesley is researching college costs. He is considering two schools: a four-year private college where tuition and fees for the current year cost about $24,000, and a four-year public university where tuition and fees for the current year cost about $10,000. Wesley learned that over the last decade, tuition and fees have increased an average of 5.6% per year in four-year private colleges and an average of 7.1% per year in four-year public colleges.

To answer Items 1–4, assume that tuition and fees continue to increase at the same average rate per year as in the last decade.

1. Complete the table of values to show the estimated tuition for the next four years.

<table>
<thead>
<tr>
<th>Years from Present</th>
<th>Private College Tuition and Fees</th>
<th>Public College Tuition and Fees</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$24,000</td>
<td>$10,000</td>
</tr>
<tr>
<td>1</td>
<td>$25,344</td>
<td>$10,710</td>
</tr>
<tr>
<td>2</td>
<td>$26,763.26</td>
<td>$11,470.41</td>
</tr>
<tr>
<td>3</td>
<td>$28,262.01</td>
<td>$12,284.81</td>
</tr>
<tr>
<td>4</td>
<td>$29,844.68</td>
<td>$13,157.03</td>
</tr>
</tbody>
</table>

2. Express regularity in repeated reasoning. Write two functions to model the data in the table above. Let \( R(t) \) represent the private tuition and fees and \( U(t) \) represent the public tuition and fees, where \( t \) is the number of years from the present.

\[
R(t) = 24,000(1.056)^t \\
U(t) = 10,000(1.071)^t
\]

3. Wesley plans to be a senior in college six years from now. Use the models above to find the estimated tuition and fees at both the private and public colleges for his senior year in college.

\[
R(6) = 33,280.88 \\
U(6) = 15,091.65
\]

4. Use appropriate tools strategically. Write an equation that can be solved to predict the number of years that it will take for the public college tuition and fees to reach the current private tuition and fees of $24,000. Find the solution using both the graphing and table features of a calculator. 10,000(1.071)^t = 24,000; graphing each side of the equation gives the intersection point (12.763, 24,000) and a table shows the values below. Therefore, it will take 13 years.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>22,776</td>
</tr>
<tr>
<td>13</td>
<td>24,393</td>
</tr>
</tbody>
</table>

MATH TIP
To solve an equation graphically on a calculator, enter each side of the equation as a separate function and find the intersection point of the two functions.

Common Core State Standards for Activity 24

HSA-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

HSA-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

HSA-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange Ohm’s law \( V = IR \) to highlight resistance \( R \).
Solving a problem like the one in Item 4 involves solving an exponential equation. An exponential equation is an equation in which the variable is in the exponent. Sometimes you can solve an exponential equation by writing both sides of the equation in terms of the same base. Then use the fact that when the bases are the same, the exponents must be equal:

\[ b^m = b^n \text{ if and only if } m = n. \]

**Example A**

Solve 6 \( \cdot 4^x = 96. \)

1. Divide both sides by 6.
2. Write both sides in terms of base 4.
3. If \( b^m = b^n, \text{ then } m = n. \)

**Example B**

Solve \( 5^2 \cdot 125^x = 125^x - 1. \)

1. Write both sides in terms of base 5.
2. Power of a Power Property: \((a^m)^n = a^{mn}\)
3. If \( b^m = b^n, \text{ then } m = n. \)
4. Solve for \( x. \)

**Try These A–B**

Solve for \( x. \) Show your work.

a. \( \frac{3}{4} - 1 = 80 \)

b. \( \frac{3}{2} = \frac{1}{32} \)

c. \( g^{3x-4} = 36^{x+1} \)

d. \( \frac{1}{7} = \frac{1}{49} \)

**Check Your Understanding**

5. When writing both sides of an equation in terms of the same base, how do you determine the base to use?

6. How could you check your solution to an exponential equation? Show how to check your answers to Try These part a.

**Lesson 24-1 Practice**

**Make use of structure.** Solve for \( x. \) by writing both sides of the equation in terms of the same base.

7. \( 2^{16} = 32 \)

8. \( 4^x - 5 = 11 \)

9. \( 2^{x-2} = 4^{x+2} \)

10. \( 8^x = \frac{1}{64} \)

11. \( 4 \cdot 5^x = 100 \)

12. \( 3 \cdot 2^{x} = 384 \)

13. \( \left( \frac{1}{3} \right)^2 = \left( \frac{5}{6} \right)^x \)

14. \( \left( \frac{1}{3} \right)^x = \left( \frac{1}{5} \right)^{3-x} \)

15. Can you apply the method used in this lesson to solve the equation \( 2^{x^2} = 27? \) Explain why or why not.

**Common Core State Standards for Activity 24 (continued)**

- HSA-REI.D.11 Explain why the x-coordinates of the points where the graphs of the equations \( y = f(x) \) and \( y = g(x) \) intersect are the solutions of the equation \( f(x) = g(x); \) find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where \( f(x) \) and/or \( g(x) \) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

- HSF-LE.A.4 For exponential models, express as a logarithm the solution to \( ab^x = d \) where \( a, c, \) and \( d \) are numbers and the base \( b \) is 2, 10, or \( e; \) evaluate the logarithm using technology.
Lesson 24-2
Solving Equations by Using Logarithms

Learning Targets:
• Solve exponential equations using logarithms.
• Estimate the solution to an exponential equation.
• Apply the compounded interest formula.

SUGGESTED LEARNING STRATEGIES: Note Taking, Group Presentation, Create Representations, Close Reading, Vocabulary Organizer

For many exponential equations, it is not possible to rewrite the equation in terms of the same base. In this case, use the concept of inverses to solve the equation symbolically.

Example A
Estimate the solution of \(3^x = 32\). Then solve to three decimal places. Estimate that \(x\) is between 3 and 4, because \(3^3 = 27\) and \(3^4 = 81\).

Step 1: \(\log_3 3^x = \log_3 32\)

Step 2: \(x \cdot \log_3 3 = \log_3 32\)

Step 3: \(x = \frac{\log_3 32}{\log_3 3}\)

Step 4: \(x \approx 3.155\)

Example B
Find the solution of \(4^{x - 2} = 35.6\) to three decimal places.

Step 1: \(\log_4 4^{x - 2} = \log_4 35.6\)

Step 2: \(x - 2 = \log_4 35.6\)

Step 3: \(x = \log_4 35.6 + 2\)

Step 4: \(x \approx 4.577\)

Try These B
Find each solution to three decimal places. Show your work.

a. \(12^{x - 2} = 240\)

b. \(4.2^{x - 4} + 0.8 = 5.7\)

c. \(e^{2x - 4} = 148\)

a. \(x = \log_{12} 240 \approx 3.079\)

b. \(x = \log_{4.2} 4.9 \approx 2.893\)

c. \(x = \ln 148 + 4 \approx 4.499\)

MATH TIP
Recall that the Inverse Properties of logarithms state that for \(b > 0, b \neq 1:\)

\[\log_b b^x = x\]

and

\[b^{\log_b x} = x\]
My Notes

Example C

If Wesley deposits the gift from his grandfather into an account that pays 4% annual interest compounded quarterly, how much money will Wesley have in the account after three years?

Substitute into the compound interest formula. Use a calculator to simplify.

\[ \begin{align*}
A &= P (1 + \frac{r}{n})^{nt} \\
&= 3000 \left(1 + \frac{0.04}{4}\right)^{4 \times 3} \\
&= 3000 \left(1 + 0.01\right)^{12} \\
&= 3000 \times 1.01^{12} \\
&= 3380.48
\end{align*} \]

Solution: Wesley will have $3380.48 in the account after three years.

Try These C

How long would it take an investment of $5000 to earn $1000 interest if it is invested in a savings account that pays 3.75% annual interest compounded monthly?

1. Rewrite the equation you wrote in Item 4 of Lesson 24-1. Then show how to solve the equation using the Inverse Property.

\[ A = 10000 (1.071)^t = 24000 \]

\[ 1.071^t = 2.4 \]

\[ \log_{10} 1.071^t = \log_{10} 2.4 \]

\[ t = \frac{\log 2.4}{\log 1.071} \]

\[ t \approx 12.763 \]

Wesley’s grandfather gave him a birthday gift of $3000 to use for college. Wesley plans to deposit the money in a savings account. Most banks pay compound interest, so he can use the formula below to find the amount of money in his savings account after a given period of time.

Compound Interest Formula

\[ A = P (1 + \frac{r}{n})^{nt} \]

where:

- \( A \) = amount in account
- \( P \) = principal invested
- \( r \) = annual interest rate as a decimal
- \( n \) = number of times per year that interest is compounded
- \( t \) = number of years

Times per Year

- Semiannually: 2
- Quarterly: 4
- Monthly: 12
- Weekly: 52
- Daily: 365

Development Math Language

As students respond to questions or discuss possible solutions to problems, monitor their use of the vocabulary compound interest and continuously compounded interest to ensure their understanding and ability to use language correctly and precisely.

Example C Create Representations, Debriefing

Compound interest problems provide students with an example of an application of exponential growth.

Differentiating Instruction

Some students may need examples in order to apply the compound interest formula correctly. Use the items below for that purpose.

a. If you deposit $500 in an account paying 3.5% annual interest compounded semiannually, how much money will be in the account after 5 years?

\[ A = 500 \left(1 + \frac{0.035}{2}\right)^{2 \times 5} \approx 594.72 \]

b. How long will it take for an investment of $2500 to earn $500 interest in an account that pays 4% annual interest compounded quarterly?

\[ \begin{align*}
A &= 2500 + 500 = 3000 \\
3000 &= 2500 \left(1 + \frac{0.04}{4}\right)^{4t} \\
1.2 &= 1.01^{4t} \\
\log_{10} 1.2 &= \log_{10} 1.01^{4t} \\
\log_{10} 1.2 &= 4t \\
t &= \frac{\log 1.2}{\log 10} \approx 4.581 \\
t &= 4 \text{ years and } 9 \text{ months}
\end{align*} \]

ACTIVITY 24: Compounding Periods and Finding Time

If students need additional help calculating compound interest, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.
Lesson 24-2
Solving Equations by Using Logarithms

Wesley’s grandfather recommends that Wesley deposit his gift into an account that earns interest compounded continuously, instead of at a fixed number of times per year.

Continuously Compounded Interest Formula
\[ A = P e^{rt} \]
- \( A \) = amount in account
- \( P \) = principal invested
- \( r \) = annual interest rate as a decimal
- \( t \) = number of years

Example D
If Wesley deposits the gift from his grandfather into an account that pays 4% annual interest compounded continuously, how much money will Wesley have in the account after three years?

Substitute into the continuously compounded interest formula. Use a calculator to simplify.

\[ A = Pe^{rt} = 3000e^{0.04(3)} \approx $3382.49 \]

Solution: Wesley will have $3382.49 in the account after three years.

Try These D
How long would it take an investment of $5000 to earn $1000 interest if it is invested in a savings account that pays 3.75% annual interest compounded continuously?

Check Your Understanding
2. How is solving exponential and logarithmic equations similar to other equations that you have solved?
3. Attend to precision. In Examples C and D, why are the answers rounded to two decimal places?
4. A bank advertises an account that pays a monthly interest rate of 0.3% compounded continuously. What value do you use for \( r \) in the continuously compounded interest formula? Explain.

LESSON 24–2 PRACTICE
Solve for \( x \) to three decimal places.
5. \( 8^x = 100 \)
6. \( 3^x = 85 \)
7. \( 3^{x+2} = 87 \)
8. \( 2^{3x-2} + 7 = 25 \)
9. \( 2 \cdot 4^x - 3 = 27 \)
10. \( e^{2x} - 1.5 = 6.7 \)
11. Make sense of problems. A deposit of $4000 is made into a savings account that pays 2.48% annual interest compounded quarterly.
   a. How much money will be in the account after three years?
   b. How long will it take for the account to earn $500 interest?
   c. How much more money will be in the account after three years if the interest is compounded continuously?

LESSON 24–2 PRACTICE
5. 2.215
6. 8.044
7. 1.367
8. 2.057
9. 0.654
10. 1.052
11. a. $4307.96
    b. 4.764 years
    c. $0.99

Activity 24 • Logarithmic and Exponential Equations and Inequalities 375
Learning Targets:
• Solve logarithmic equations.
• Identify extraneous solutions to logarithmic equations.
• Use properties of logarithms to rewrite logarithmic expressions.

SUGGESTED LEARNING STRATEGIES: Create Representations, Vocabulary Organizer, Note Taking, Group Presentation

Equations that involve logarithms of variable expressions are called logarithmic equations. You can solve some logarithmic equations symbolically by using the concept of functions and their inverses. Since the domain of logarithmic functions is restricted to the positive real numbers, it is necessary to check for extraneous solutions when solving logarithmic equations.

An extraneous solution is a solution that arises from a simplified form of the equation that does not make the original equation true.

Example A
Solve \( \log_4 (3x - 1) = 2 \).

Step 1: \( 4^{\log_4 (3x - 1)} = 4^2 \) Write in exponential form using 4 as the base.

Step 2: \( 3x - 1 = 16 \) Use the Inverse Property to simplify the left side.

Step 3: \( x = \frac{17}{3} \) Solve for \( x \).

Check: \( \log_4 (3 \cdot \frac{17}{3} - 1) = \log_4 16 = 2 \)

Try These A
Solve for \( x \). Show your work.

a. \( \log_2 (x - 1) = 5 \)

b. \( \log_3 (2x - 3) = 3 \)

c. \( 4 \ln (3x) = 8 \)

To solve other logarithmic equations, use the fact that when the bases are the same, \( m > 0, n > 0, \) and \( b = 1 \), the logarithmic values must be equal:

\( \log_a m = \log_a n \) if and only if \( m = n \)
Example B
Solve \( \log_3 (2x - 3) = \log_3 (x + 4) \).

Step 1: \( 2x - 3 = x + 4 \)  
If \( \log_b m = \log_b n \), then \( m = n \).

Step 2: \( x = 7 \)  
Solve for \( x \).

Check: \( \log_3 (2 \cdot 7 - 3) = \log_3 (7 + 4) \)  
\( \log_3 11 = \log_3 11 \)

Try These B
Solve for \( x \). Check for extraneous solutions. Show your work.

a. \( \log_6 (3x + 4) = 1 \)  
b. \( \log_5 (7x - 2) = \log_5 (3x + 6) \)
c. \( \ln 10 - \ln (4x - 6) = 0 \)

Sometimes it is necessary to use properties of logarithms to simplify one side of a logarithmic equation before solving the equation.

Example C
Solve \( \log_2 x + \log_2 (x + 2) = 3 \).

Step 1: \( \log_2 [x(x + 2)] = 3 \)  
Product Property of Logarithms

Step 2: \( 2^{\log_2 [x(x + 2)]} = 2^3 \)  
Write in exponential form using 2 as the base.

Step 3: \( x(x + 2) = 8 \)  
Use the Inverse Property to simplify.

Step 4: \( x^2 + 2x - 8 = 0 \)  
Write as a quadratic equation.

Step 5: \( (x + 4)(x - 2) = 0 \)  
Solve the quadratic equation.

Step 6: \( x = -4 \) or \( x = 2 \)  
Check for extraneous solutions.

Check: \( \log_2 (-4) + \log_2 (-4 + 2) = \frac{3}{2} \)  
\( \log_2 (4) + \log_2 (-2) = 3 \)  
\( \log_2 2 + \log_2 (2 + 2) = \frac{5}{2} \)  
\( \log_2 2 + \log_2 4 = 3 \)  
\( \log_2 8 = 3 \)  
\( 3 = 3 \)

Because \( \log_2 (-4) \) and \( \log (-2) \) are not defined, \(-4\) is not a solution of the original equation; thus it is extraneous.

The solution is \( x = 2 \).

Try These C
Solve for \( x \), rounding to three decimal places if necessary. Check for extraneous solutions.

a. \( \log_4 (x + 6) - \log_4 x = 2 \)
b. \( \ln (2x + 2) + \ln 5 = 2 \)
c. \( \log_6 2x + \log_6 (x - 3) = 3 \)  
\( 4 \); \(-1\) is extraneous
transitive property of equality to state that the solution of this set of equations is the solution of the original equation.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand the concept of extraneous solutions for logarithmic equations. If students struggle with the concept of extraneous solutions for logarithmic equations, demonstrate the parallel with extraneous solutions of radical equations.

ASSESS

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students’ answers to the Lesson Practice to ensure that they understand how to solve logarithmic equations. Students who need additional practice approximating solutions can apply graphing techniques to solve linear or quadratic equations. Provide students with examples that they can solve algebraically. Guide them to approximate the solutions graphically and then to confirm the approximation by finding the solution algebraically.

Answers

1. Sample answer: Solving a logarithmic equation can require simplifying expressions to a quadratic equation having two solutions. Because the input of a logarithm must be a positive real number, one solution may not satisfy the original equation.
2. No; Than needs to check both solutions. A solution is extraneous only if it causes a logarithm to have an input that is not positive.

Check Your Understanding

1. Explain how it is possible to have more than one solution to a simplified logarithmic equation, only one of which is valid.
2. Critique the reasoning of others. Than solves a logarithmic equation and gets two possible solutions, −2 and 4. Than immediately decides that −2 is an extraneous solution, because it is negative. Do you agree with his decision? Explain your reasoning.

LESSON 24-3 PRACTICE

Solve for x, rounding to three decimal places if necessary. Check for extraneous solutions.

3. \( \log_2 (3x + 4) = 2 \)
4. \( \log_3 (4x + 1) = 4 \)
5. \( \log_{12} (4x − 2) = \log_{12} (x + 10) \)
6. \( \log_3 3 + \log_2 (x − 4) = 4 \)
7. \( \ln (x + 4) − \ln (x − 4) = 4 \)
8. Construct viable arguments. You saw in this lesson that logarithmic equations may have extraneous solutions. Do exponential equations ever have extraneous solutions? Justify your answer.
### Lesson 24-4
#### Exponential and Logarithmic Inequalities

**Learning Targets:**
- Solve exponential inequalities.
- Solve logarithmic inequalities.

**Suggested Learning Strategies:** Note Taking, Group Presentation, Create Representations

You can use a graphing calculator to solve exponential and logarithmic inequalities.

**Example A**

Use a graphing calculator to solve the inequality $4.2^{x+3} > 9$.

**Step 1:** Enter $4.2^{x+3}$ for Y1 and 9 for Y2.

**Step 2:** Find the x-coordinate of the point of intersection: $x \approx -1.469$.

**Step 3:** The graph of $y = 4.2^{x+3}$ is above the graph of $y = 9$ when $x > -1.469$.

**Solution:** $x > -1.469$

**Try These A**

Use a graphing calculator to solve each inequality.

- a. $3 \cdot 5^{1-x} < 75, x > -0.978$
- b. $\log 10x \geq 1.5, x \geq 3.162$
- c. $7.2 \ln x + 39 < 12, 0 < x < 3.080$

**Example B**

Scientists have found a relationship between atmospheric pressure and altitudes up to 50 miles above sea level that can be modeled by $P = 14.7(0.5)^{\frac{x}{36}}$. $P$ is the atmospheric pressure in lb/in.$^2$. Solve the equation $P = 14.7(0.5)^{\frac{x}{36}}$ for $a$. Use this equation to find the atmospheric pressure when the altitude is greater than 2 miles.

**Step 1:** Solve the equation for $a$.

$$P = 14.7 \cdot 0.5^{\frac{x}{36}}$$

Divide both sides by 14.7.

$$\log_{0.5} \left( \frac{P}{14.7} \right) = \log_{0.5} 0.5^{\frac{x}{36}}$$

Take the log base 0.5 of each side.

$$\log_{0.5} \left( \frac{P}{14.7} \right) = \frac{a}{36}$$

Simplify.

$$3.6 \log_{0.5} \left( \frac{P}{14.7} \right) = a$$

Multiply both sides by 3.6.

$$\frac{3.6\log_{0.5} P}{\log_{0.5} 0.5} = a$$

Use the Change of Base Formula.

**Example A** Create Representations, Think-Pair-Share, Group Presentation

Students solve the inequality $14.7(0.5)^{\frac{x}{36}} < 5$ by setting each side of the equation equal to $y$ and then graphing these equations, finding the intersection, and selecting the interval where the exponential function is below the linear function. There is also a restriction that the model holds for $x < 50$, so the solution is $5.601 < x < 50$.

**Technology Tip**

In order to use the graphing feature of their calculators to approximate equation solutions, students must define each function that the calculator will graph. These functions are then available for use with the table feature. For additional technology resources, visit SpringBoard Digital.
Step 2: Use your graphing calculator to solve the inequality

\[
\frac{3.6 \log \left( \frac{P}{14.7} \right)}{\log 0.5} > 2.
\]

The graph of \( y = \frac{3.6 \log \left( \frac{x}{14.7} \right)}{\log 0.5} \) is above the graph of \( y = 2 \) when

\( 0 < x < 10.002 \).

Solution: When the altitude is greater than 2, the atmospheric pressure is between 0 and 10.002 lb/in.².

Try These B

Suppose that the relationship between \( C \), the number of digital cameras supplied, and the price \( x \) per camera in dollars is modeled by the function

\[ C = -400 + 180 \log x. \]

a. Find the range in the price predicted by the model if there are between 20 and 30 cameras supplied.

b. Solve the equation for \( x \). Use this equation to find the number of cameras supplied when the price per camera is more than $300.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand the methods used to solve exponential and logarithmic inequalities. Invite students to share their responses and discuss their favorite strategies for solving inequalities.

Answers
2. Sample answer: The answer is a range of values instead of a single value.
3. Sample answer: Graph both sides of the inequality. When one graph is above the other graph, then that side of the inequality is greater than the other side. This will occur on one side of the intersection point. Find the \( x \)-value of the intersection point and use the appropriate inequality symbol(s) to write the solution.

Check Your Understanding

2. How are exponential and logarithmic inequalities different from exponential and logarithmic equations?
3. Describe how to find the solution of an exponential or logarithmic inequality from a graph. What is the importance of the intersection point in this process?

LESSON 24-4 PRACTICE

Use a graphing calculator to solve each inequality.

4. \( 16.4(0.87)^{x-15} \geq 10 \)
5. \( 30 < 25 \log (3.5x - 4) + 12.6 < 50 \)
6. \( 4.5e^x \leq 2 \)
7. \( \ln (x - 7.2) > 1.35 \)

ADAPT

Check students’ answers to the Lesson Practice to ensure that they understand how to solve exponential and logarithmic inequalities. Students who need additional practice may benefit from applying the techniques of this lesson to review solving linear or quadratic inequalities.
Logarithmic and Exponential Equations and Inequalities

ACTIVITY 24 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 24-1
1. Which exponential equation can be solved by rewriting both sides in terms of the same base?
   - A. \(4^x = 12\)
   - B. \(6 \cdot 2^{-3} = 256\)
   - C. \(3^{x+2} = 22\)
   - D. \(e^x = 58\)

2. Solve for \(x\).
   - a. \(16^x = 32\)
   - b. \(8 \cdot 3^x = 216\)
   - c. \(5^x = \frac{1}{625}\)
   - d. \(7^{2x} = 343^{x-1}\)
   - e. \(4^x + 8 = 72\)
   - f. \(e^x = 3\)
   - g. \(e^{2x} = 2\)
   - h. \(3e^{2x} = 42\)

Lesson 24-2
3. Solve for \(x\) to three decimal places.
   - a. \(7^x = 300\)
   - b. \(5^{-x} = 135\)
   - c. \(3^{x+1} = 5 = 80\)
   - d. \(3 \cdot 6^x = 0.01\)
   - e. \(5^x = 212\)
   - f. \(3(2^{x+1}) = 350\)

4. A deposit of $1000 is made into a savings account that pays 4% annual interest compounded monthly.
   - a. How much money will be in the account after 6 years?
   - b. How long will it take for the $1000 to double?

5. June invests $7500 at 12% interest for one year.
   - a. How much would she have if the interest is compounded yearly?
   - b. How much would she have if the interest is compounded daily?

6. June invests $7500 at 12% interest for one year.
   - a. How much would she have if the interest is compounded yearly?
   - b. How much would she have if the interest is compounded daily?

7. If $4000 is invested at 7% interest per year compounded continuously, how long will it take to double the original investment?

8. At what annual interest rate, compounded continuously, will money triple in nine years?
   - A. 1.3\%
   - B. 7.3\%
   - C. 8.1\%
   - D. 12.2\%

Lesson 24-3
9. Solve for \(x\). Check for extraneous solutions.
   - a. \(\log_2 (5x - 2) = 3\)
   - b. \(\log_2 (2x - 3) = 2\)
   - c. \(\log_3 (5x + 3) = \log_3 (3x + 11)\)
   - d. \(\log_4 (x + 2) = 1\)
   - e. \(\log_5 (x + 8) = 2 - \log_5 (x)\)
   - f. \(\log_6 (x + 6) - \log_6 (x) = 3\)
   - g. \(\log_7 (x + 3) - \log_7 (x + 5) = \log_7 10\)
   - h. \(\ln 3x = 40\)
   - i. \(\ln 4x = 30\)
   - j. \(\log_3 (x + 8) = 2 - \log_3 (x)\)
### ACTIVITY 24 Continued

**10.** a. The solutions for \( \log(x - 2) \) must be greater than 2, because any solutions equal to or less than 2 will result in the log of a negative number.

b. The solutions for \( \log(x + 3) \) must be greater than −3, because any solutions equal to or less than −3 will result in the log of a negative number.

**11.** Solve for \( x \) to three decimal places using a graphing calculator.

a. \( \ln 3x = x^2 - 2 \)

b. \( \log (x + 7) = x^2 - 6x + 5 \)

**Lesson 24-4**

**12.** Use a graphing calculator to solve each inequality.

a. \( 2000 < 1500(1.04)^{12} < 3000 \)

b. \( 4.5 \log (2x) + 8.4 \geq 9.2 \)

c. \( \log_2 (2x) \leq \log_3 (x + 3) \)

d. \( \log_2 2x \leq \log_4 (x + 3) \)

e. \( 5x + 3 \leq 2x + 4 \)

**MATHEMATICAL PRACTICES**

**Look For and Make Use of Structure**

13. Explore how the compounded interest formula is related to the continuously compounded interest formula.

a. Consider the expression \( \left(1 + \frac{1}{m}\right)^n \), where \( m \) is a positive integer. Enter the expression in your calculator as \( y_1 \). Then find the value of \( y_1(1000) \), \( y_1(10,000) \), and \( y_1(1,000,000) \).

b. As \( m \) increases, what happens to the value of the expression?

c. The compounded interest formula is \( A = P \left(1 + \frac{1}{m}\right)^n \). Let \( m = \frac{r}{n} \). Explain why the formula may be written as \( A = P \left(1 + \frac{r}{n}\right)^n \).

d. As the number of compounding periods, \( n \), increases, so does the value of \( m \). Explain how your results from parts b and c show the connection between the compounded interest formula and the continuously compounded interest formula.

**ADDITIONAL PRACTICE**

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.
Exponential and Logarithmic Equations

EVALUATING YOUR INTEREST

1. **Make use of structure.** Express each exponential statement as a logarithmic statement.
   - a. \(5^3 = \frac{1}{125}\)
   - b. \(7^2 = 49\)
   - c. \(20^2 = 400\)
   - d. \(3^8 = 729\)

2. **Express each logarithmic statement as an exponential statement.**
   - a. \(\log_5 512 = 3\)
   - b. \(\log_{\frac{1}{125}} 125 = -3\)
   - c. \(\log_6 64 = 6\)
   - d. \(\log_{10} 14,641 = 4\)

3. **Evaluate each expression without using a calculator.**
   - a. \(25^{\log_{10} x}\)
   - b. \(\log_3 3^7\)
   - c. \(\log_8 27\)
   - d. \(\log_5 1\)
   - e. \(\log_4 32 - \log_4 5\)
   - f. \(\log_{10} 25\)

4. **Solve each equation symbolically.** Give approximate answers rounded to three decimal places. Check your solutions. Show your work.
   - a. \(4^{2x - 1} = 64\)
   - b. \(5^x = 38\)
   - c. \(3^{x+2} = 98.7\)
   - d. \(2^{3x+4} + 7.5 = 23.6\)
   - e. \(\log_3 (2x + 1) = 4\)
   - f. \(\log_6 (3x - 2) = \log_6 (x + 1)\)
   - g. \(\log_5 (3x - 2) + \log_5 8 = 5\)
   - h. \(\log_6 (x - 5) + \log_6 x = 2\)

5. Let \(f(x) = \log_2 (x - 1) + 3\).
   - a. Sketch a parent graph and a series of transformations that result in the graph of \(f\).
   - b. Give the equation of the vertical asymptote of the graph of \(f\).

6. **Make sense of problems.** Katie deposits \$10,000 in a savings account that pays 8.5% interest per year, compounded quarterly. She does not deposit more money and does not withdraw any money.
   - a. Write the formula to find the amount in the account after 3 years.
   - b. Find the total amount she will have in the account after 3 years.

7. **Solve logarithmic equations.** How long would it take an investment of \$6500 to earn \$1200 interest if it is invested in a savings account that pays 4% annual interest compounded quarterly? Show the solution both graphically and symbolically.

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**Common Core State Standards for Embedded Assessment 3**

- **HSA-CED.A.1**: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.
- **HSA-CED.A.2**: Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
- **HSF-IF.C.7**: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.
- **HSF-IF.C.7e**: Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
- **HSF-LE.A.4**: For exponential models, express as a logarithm the solution to \(ab^x = d\) where \(a\), \(b\), and \(d\) are numbers and the base \(b\) is 2, 10, or \(e\); evaluate the logarithm using technology.

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**Embedded Assessment 3**

**Assessment Focus**
- Solving exponential equations
- Solving logarithmic equations
- Solving real-world applications of exponential and logarithmic functions

**Answer Key**

1. a. \(\log_2 \left(\frac{1}{125}\right) = -3\)
   - b. \(\log_3 49 = 2\)
   - c. \(\log_{10} 400 = 2\)
   - d. \(\log_8 729 = 6\)

2. a. \(8^3 = 512\)
   - b. \(9^{-3} = \frac{1}{729}\)
   - c. \(2^{10} = 1024\)
   - d. \(11^{\frac{3}{2}} = 14.641\)

3. a. \(x = 8\)
   - b. \(x = 2\)
   - c. \(3\)
   - d. \(0\)
   - e. \(3\)
   - f. \(2\)

4. a. \(2\)
   - b. \(2.260\)
   - c. \(2.180\)
   - d. \(2.670\)
   - e. \(40\)
   - f. \(1.5\)
   - g. \(2\)
   - h. \(9\)

5. a. \(f(x) = \log_2 (x - 1) + 3\)

6. a. \(A = 10,000\left(1 + \frac{0.085}{4}\right)^{4(3)}\)
   - b. \$12,870.19

7. 4.256 years; Check students’ graphs. Solution should appear as point of intersection of \(f(x) = 1.010^x\) and \(f(x) = 1.1846\).
### Embedded Assessment 3

**Use after Activity 24**

### Exponential and Logarithmic Equations

#### EVALUATING YOUR INTEREST

**Scoring Guide**

<table>
<thead>
<tr>
<th>Mathematics Knowledge and Thinking (Items 1–7)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Fluency and accuracy in evaluating and rewriting exponential and logarithmic equations and expressions</td>
<td>• Largely correct work when evaluating and rewriting exponential and logarithmic equations and expressions</td>
<td>• Difficulty when evaluating and rewriting logarithmic and exponential equations and expressions</td>
<td>• Mostly inaccurate or incomplete work when evaluating and rewriting logarithmic and exponential equations and expressions</td>
<td></td>
</tr>
<tr>
<td>• Effective understanding of and accuracy in solving logarithmic and exponential equations algebraically and graphically</td>
<td>• Adequate understanding of how to solve logarithmic and exponential equations algebraically and graphically leading to solutions that are usually correct</td>
<td>• Partial understanding of how to solve logarithmic and exponential equations algebraically and graphically</td>
<td>• Inaccurate or incomplete understanding of how to solve exponential and logarithmic equations algebraically and graphically</td>
<td></td>
</tr>
<tr>
<td>• Effective understanding of logarithmic functions and their key features as transformations of a parent graph</td>
<td>• Adequate understanding of logarithmic functions and their key features as transformations of a parent graph</td>
<td>• Partial understanding of logarithmic functions and their key features as transformations of a parent graph</td>
<td>• Little or no understanding of logarithmic functions and their key features as transformations of a parent graph</td>
<td></td>
</tr>
</tbody>
</table>

**Problem Solving (Items 6, 7)**

| • An appropriate and efficient strategy that results in a correct answer | • A strategy that may include unnecessary steps but results in a correct answer | • A strategy that results in some incorrect answers | • No clear strategy when solving problems |

**Mathematical Modeling / Representations (Items 5–7)**

| • Fluency in modeling a real-world scenario with an exponential equation or graph | • Little difficulty in accurately modeling a real-world scenario with an exponential equation or graph | • Some difficulty in modeling a real-world scenario with an exponential equation or graph | • Significant difficulty with modeling a real-world scenario with an exponential equation or graph |
| • Effective understanding of how to graph a logarithmic function using transformations | • Largely correct understanding of how to graph a logarithmic function using transformations | • Partial understanding of how to graph a logarithmic function using transformations | • Mostly inaccurate or incomplete understanding of how to graph a logarithmic function using transformations |

**Reasoning and Communication (Items 6, 7)**

| • Clear and accurate use of mathematical work to justify an answer | • Correct use of mathematical work to justify an answer | • Partially correct justification of an answer using mathematical work | • Incorrect or incomplete justification of an answer using mathematical work |

#### Common Core State Standards for Embedded Assessment 3 (cont.)

- **HSF-BF.A.1** Write a function that describes a relationship between two quantities.*
- **HSF-BF.A.1c (+)** Compose functions.
- **HSF-BF.B.3** Identify the effect on the graph of replacing $f(x)$ by $f(x) + k, kf(x), f(kx)$, and $f(x + k)$ for specific values of $k$ (both positive and negative); find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
- **HSF-BF.B.4** Find inverse functions.
- **HSF-BF.B.4b (+)** Verify by composition that one function is the inverse of another.